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Reflection by semi-infinite diffusers

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Summary. — *Total and directional reflectances are derived and tabulated for semi-infinite diffusers with matrices of various refractive indices and scattering according to the phase function $\varpi_0(1 + x \cos \theta)$.*

Sommaire. — *On établit les formules algébriques et on donne des tables des valeurs numériques des réflectances totales et directionnelles, calculées pour des diffuseurs semi-infinis caractérisés par des indices et des coefficients de diffusion satisfaisant à la loi $\varpi_0 (1 + x \cos \theta)$.*

Zusammenfassung. — *Der Gesamwert und der gerichtete Anteil der Rückstrahlung eines den unendlichen Halbraum erfüllenden streuenden Mediums werden für Matrizzen von verschiedenen Brechwerten und einer Streuung gemäss der Phasenfunktion $\varpi_0 (1 + x \cos \theta)$ berechnet und tabelliert.*

I. **Introduction.** — There are potentially at least two major applications of a precise knowledge of the total and directional reflectances of diffusing materials, namely colour prediction and spectro-photometric chemical analysis.

A number of attempts have been made to describe the reflectance and transmittance of plane parallel diffusers in terms of the scattering and absorbing properties of their constituents ; by Kubelka and Munk [1], Amy [2] and Kubelka [3] using the Schwarzchild approximation in solving the relevant equations of radiative transfer, and by others. Pitts [4] has studied the light scattered in photographic emulsions using the Eddington and Mark approximate methods. Exact methods have been developed by Chandrasekhar [5] for obtaining some of the optical properties of diffusers, including the reflected intensity distribution for collimated light incident on semi-infinite diffusers scattering isotropically, according to a phase function $\varpi_0(1 + x \cos \theta)$ or to the Rayleigh phase function ; the two former cases include absorption within the medium, and tables of some of the relevant H-functions needed in evaluating the solutions have been given for isotropic and Rayleigh phase functions, and for the phase function $\varpi_0(1 + \cos \theta)$.

The above mentioned researches have discussed primarily diffusers whose matrices have unit refractive index. Whenever allowance has been made for matrices of higher refractive index, in which case surface reflections occur internally, the problem has been simplified by assuming a reflection coefficient independent of direction ; i. e. the surface has been assumed perfectly diffusing for internally incident radiation reflected and transmitted through it.

The present paper discusses total and directional reflectances of semi-infinite diffusers of constant composition, scattering according to a phase function of the form $\varpi_0(1 + x \cos \theta)$. Part I deals with media with N (matrix) = 1. Part II deals with media in which N (matrix) exceeds unity. Throughout, an accuracy has been sought better than at present obtainable in best measurements, i. e. of the order of $\pm$ 0.001 or better in reflectance.

Part I. Matrices of Refractive Index Unity

II. **Isotropic scattering.** — Let a collimated beam of radiation of flux $\pi F_i(\mu_0)$ per unit area normal to the beam be incident on the surface of a semi-infinite diffuser at an angle to the outward normal whose cosine is $-\mu_0$ Then Chandrasekhar [5], (§ 26) has shown that for isotropic scattering the reflected intensity in a direction μ is

$$(2.1) \quad I_r(\mu) = \frac{\varpi_0}{4} F_i(\mu_0) \frac{\mu_0}{\mu + \mu_0} H(\mu) H(\mu_0)$$

where ϖ_0, the albedo for single scattering, is defined by

$$(2.2) \quad \varpi_0 = \frac{\sigma}{\sigma + \alpha},$$

σ and α being the scattering and absorption coeffi-

cients respectively, while $H(\mu)$ is a function defined by the equation

$$(2.3)\quad H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} H(\mu')\, d\mu'.$$

For isotropic scattering, $\Psi(\mu') = \frac{1}{2}$, and the function $H(\mu)$ has been tabulated by CHANDRASEKHAR [5], (Table XI).

The reflected flux per unit area of surface is

$$(2.4)\quad \pi F_r = \int I_r(\mu)\, \mu\, d\Omega,$$

$d\Omega$ being an element of solid angle, while the total reflectance for light incident in a direction μ_0 is given by

$$(2.5)\quad R(\mu_0) = \frac{\pi F_r}{\pi F_i(\mu_0)\mu_0}.$$

It follows readily from the properties of H- functions that

$$(2.6)\quad R(\mu_0) = 1 - H(\mu_0)\,(1 - \varpi_0)^{1/2}.$$

If the incident radiation is uniformly diffused, with flux $f\, d\Omega$ per unit area normal to the beam in an element of solid angle $d\Omega$, then from (2.1) the reflected intensity in a direction μ is

$$I_r(\mu) = \frac{1}{2} \varpi_0 f H(\mu) \int_0^1 \frac{\mu_0 H(\mu_0)}{\mu + \mu_0} d\mu_0$$

$$(2.7)\quad = f[1 - H(\mu)\,(1 - \varpi_0)^{1/2}],$$

a result which could also have been obtained directly from (2.6) from a well known photometric reciprocity law.

The total reflected flux per unit area of surface is

$$\pi F_r = 2\pi f \int_0^1 \mu\,[1 - H(\mu)\,(1 - \varpi_0)^{1/2}]\, d\mu$$

$$(2.8)\quad = \pi f - 2\pi f (1 - \varpi_0)^{1/2} \int_0^1 \mu H(\mu)\, d\mu.$$

The reflectance for totally diffused incident radiation is thus

$$(2.9)\quad R_D = 1 - 2(1 - \varpi_0)^{1/2} \int_0^1 \mu H(\mu)\, d\mu.$$

The value of the integral, the first moment of $H(\mu)$, is tabulated by CHANDRASEKHAR [5], (Table XXXIII).

III. **Scattering according to a phase function** $\varpi_0(1 + x\cos\theta)$. — Scattering by real media rarely occurs isotropically. The angular distribution of scattered light may be described by the phase function $p(\cos\theta)$, where $p(\cos\theta)\, d\Omega/4\pi\varpi_0$ is the fraction of scattered radiation entering solid angle $d\Omega$: thus

$$(3.1)\quad \int p(\cos\theta) \frac{d\Omega}{4\pi} = \varpi_0.$$

For a phase function $\varpi_0(1 + x\cos\theta)$, the reflected intensity $I_r(\mu, \varphi)$ of a semi-infinite medium in direction μ, azimuth φ, is given by CHANDRASEKHAR [5], (§ 46)

$$(3.2)\quad I_r(\mu,\varphi) = \frac{1}{4} \varpi_0 F_i(\mu_0) \times$$
$$\left\{ H(\mu) H(\mu_0)\,[1 - c(\mu + \mu_0) - x(1 - \varpi_0)\,\mu\mu_0] + x(1 - \mu^2)^{1/2}(1 - \mu_0^2)^{1/2} H^{(1)}(\mu)\, H^{(1)}(\mu_0) \cos(\varphi_0 - \varphi) \right\}$$
$$\times \frac{\mu_0}{\mu + \mu_0},$$

the direction and azimuth angle of the incident collimated beam being $-\mu_0$ and φ_0 respectively. $H(\mu)$ and $H^{(1)}(\mu)$ are H-functions defined by (2.3), with

$$\Psi(\mu) = \frac{1}{2} \varpi_0\,[1 + x(1 - \varpi_0)\,\mu^2]$$

and

$$\Psi(\mu) = \frac{1}{4} x\, \varpi_0 (1 - \mu^2)$$

respectively, while

$$c = x(1 - \varpi_0)\,\varpi_0 \frac{\alpha_1}{2 - \varpi_0 \alpha_0},$$

α_0 and α_1 being the moments of $H(\mu)$ of order zero and unity :

$$\alpha_0 = \int_0^1 H(\mu)\, d\mu, \quad \alpha_1 = \int_0^1 H(\mu)\, \mu\, d\mu.$$

Integration around φ yields for the element of reflected flux in $d\mu$ per unit area of surface

$$\pi\, dF_r = \frac{\pi \varpi_0 F_i(\mu_0)\, \mu\mu_0}{2(\mu + \mu_0)} H(\mu)\, H(\mu_0) \times$$
$$[1 - c(\mu + \mu_0) - x(1 - \varpi_0)\,\mu\mu_0]\, d\mu_0.$$

Hence the total reflected flux is

$$(3.3)\quad \pi F_r =$$
$$= \pi F_i(\mu_0)\,\mu_0 \left[1 - H(\mu_0) \left\{ 1 - \frac{\varpi_0}{2} (\alpha_0 - c\,\alpha_1) \right\} \right],$$

and the total reflectance for light incident in a direction μ_0 is

$$(3.4)\quad R(\mu_0) = 1 - H(\mu_0) \left\{ 1 - \frac{\varpi_0}{2} (\alpha_0 - c\,\alpha_1) \right\}.$$

If the incident radiation is uniformly diffused, the reflected flux per unit area is

$$(3.5)\quad \pi F_r = \int R(\mu_0)\, \mu_0\, f\, d\Omega =$$
$$= \pi f \left[1 - 2\alpha_1 \left\{ 1 - \frac{\varpi_0}{2} (\alpha_0 - c\,\alpha_1) \right\} \right].$$

Thus the reflectance for totally diffused incident radiation is

$$(3.6) \quad R_D = 1 - 2\alpha_1 \left\{ 1 - \frac{\varpi_0}{2}(\alpha_0 - c\,\alpha_1) \right\}.$$

Values of the relevant H-functions, of the moments α_0 and α_1 and of the constant c are given by CHANDRASEKHAR [5], (Tables XVI and XVII) for $x = 1$.

IV. **Scattering according to a phase function $\varpi_0(1 + x \cos \theta)$; Eddington's approximation.** — The exact results given in § III lead to numerical values for the reflectance provided tables of relevant H-functions are available ; these are only at present available for the cases $x = 1$ and $x = 0$ (§ II). Using EDDINGTON's approximation, however, reflectances may readily be obtained for any value of x. While these differ slightly from the exact reflectances for $x = 0$ and $x = 1$, they can be used to provide very good interpolations for other values of x.

PITTS [4] has discussed the behaviour of plane parallel diffusing media for light incident normally on the surface, and has shown that if the phase function be expanded in a series of LEGENDRE polynomials, the total intensity is everywhere independent of terms of the second and higher degree, on the EDDINGTON approximation. Thus to this approximation the phase function is fully described by $\varpi_0(1 + x \cos \theta)$.

PITTS's discussion can be extended in a straightforward manner to the case of radiation incident on a semi-infinite diffuser in a direction μ_0, and it can be shown that the total reflectance is

$$(4.1) \quad R(\mu_0) = \frac{\varpi_0}{2\sqrt{\chi} + 3 - \varpi_0 x} \left\{ -x + \frac{3 + (1 - \varpi_0)x}{1 + \mu_0\sqrt{\chi}} \right\}$$

where

$$\chi = (3 - \varpi_0 x)(1 - \varpi_0).$$

The reflectance for totally diffused incident radiation follows readily by integration

$$(4.2) \quad R_D = \frac{\varpi_0}{2\sqrt{\chi} + 3 - \varpi_0 x} \left\{ -x + \frac{6 + 2(1 - \varpi_0)x}{\chi} \left[\sqrt{\chi} - \log(1 + \sqrt{\chi}) \right] \right\}.$$

In the case of isotropic scattering, $x = 0$ and (4.1) and (4.2) simplify to

$$(4.3) \quad R(\mu_0) = \frac{\varpi_0}{\left(1 + 2\sqrt{\frac{\lambda}{3}}\right)(1 + \mu_0\sqrt{3\lambda})}$$

and

$$(4.4) \quad R_D = \frac{2\varpi_0[\sqrt{3\lambda} - \log(1 + \sqrt{3\lambda})]}{\lambda(3 + 2\sqrt{3\lambda})}$$

where $\lambda = 1 - \varpi_0$.

TABLE 1

Total Reflectances for isotropic scattering and for a phase function ϖ_0 $(1 + \cos \theta)$

ϖ_0	$\mu_0 = 1$		Diffused incident Radiation	
	Isotropic	$\varpi_0(1+\cos\theta)$	Isotropic	$\varpi_0(1+\cos\theta)$
1.000	1.00000	1.00000	1.00000	1.00000
0.999	0.91285	0.89367	0.92971	0.91446
0.995	0.81705	0.77877	0.84985	0.81945
0.990	0.75275	0.70270	0.79457	0.75482
0.975	0.64092	0.57344	0.69501	0.64140
0.950	0.53555	0.45552	0.59667	0.53311
0.925	0.46655	0.38104	0.52965	0.46172
0.900	0.41495	0.32712	0.47802	0.40825
0.85	0.33966	—	0.40017	—
0.80	0.28526	0.20015	0.34187	0.27406
0.7	0.20867	0.13286	0.25655	0.19626
0.6	0.15541	0.09065	0.19471	0.14318
0.5	0.11521	0.06192	0.14653	0.10411
0.4	0.08336	0.04147	0.10934	0.07394
0.3	0.05721	0.02638	0.07445	0.04986
0.2	0.03524	0.01513	0.04626	0.03018
0.1	0.01639	0.00649	0.02170	0.01382
0	0.00000	0.00000	0.00000	0.00000

V. **Results.** — Table 1 contains data, computed from the exact results of sections II and III, on the total reflectances for radiation incident along the normal or perfectly diffused, for the cases of isotropic scattering and scattering according to the phase function $\varpi_0(1 + \cos \theta)$. Results are given for values of ϖ_0 tabulated by CHANDRASEKHAR, and for some additional values of ϖ_0 close to unity for which supplementary H-functions (not obtainable to sufficient accuracy by interpolation) have been computed.

TABLE 2

Reflectances obtained with the EDDINGTON approximation for isotropic scattering and for a phase function $\varpi_0(1 + \cos \theta)$

$\mu_0 = 1$

ϖ_0	Isotropic scattering		$\varpi_0(1 + \cos\theta)$	
	$R(1)$	Correction	$R(1)$	Correction
1.000	1.00000	0.00000	1.00000	0.00000
0.999	0.91376	—0.00091	0.89484	—0.00117
0.995	0.81952	—0.00247	0.78125	—0.00247
0.990	0.75649	—0.00374	0.70628	—0.00358
0.975	0.64722	—0.00630	0.57885	—0.00541
0.950	0.54426	—0.00871	0.46270	—0.00718
0.925	0.47666	—0.01011	0.38922	—0.00818
0.900	0.42596	—0.01101	0.33590	—0.00878
0.85	0.35152	—0.01186	0.26104	—
0.80	0.29729	—0.01203	0.20957	—0.00942
0.7	0.22005	—0.01138	0.14183	—0.00897
0.6	0.16548	—0.01007	0.09873	—0.00808
0.5	0.12372	—0.00851	0.06888	—0.00696
0.4	0.09017	—0.00681	0.04713	—0.00566
0.3	0.06230	—0.00511	0.03072	—0.00434
0.2	0.03860	—0.00336	0.01802	—0.00289
0.1	0.01806	—0.00167	0.00801	—0.00152
0	0.00000	0.00000	0.00000	0.00000

Table 2 lists total reflectances, computed for normal incidence from the approximate equation (4.1), for $x = 0$ (isotropic scattering) and $x = 1$. The correc-

tions to the two latter sets of results may be derived by comparison with columns 2 and 3 of Table 1, and are listed in Table 2, from which it will be seen that the two sets of corrections are close together, differing at most by little more than 0.002. Consequently the corrections to results derived from (4.1) for intermediate values of x can be obtained reliably by linear interpolation. In Table 3 total reflectances for normally incident radiation obtained in this way are listed for $x = 0$, 0.25, 0.50, 0.75 and 1.00; reflectances for intermediate values of x may be obtained correct to one unit in the third decimal place by linear interpolation. The reliability of the above procedure has been checked by evaluating a few H-functions for $x = 0.5$ and comparing the reflectances so derived with the corrected values.

Table 3

Reflectance for a phase function $\varpi_0\ (1 + x \cos \theta)$

$\mu_0 = 1$

ϖ_0	x				
	0	0.25	0.5	0.75	1.00
1.000	1.00000	1.00000	1.00000	1.00000	1.00000
0.999	0.91285	0.9090	0.9046	0.8996	0.89367
0.995	0.81705	0.8094	0.8006	0.7906	0.77877
0.990	0.75275	0.7427	0.7312	0.7180	0.70270
0.975	0.64092	0.6273	0.6116	0.5940	0.57344
0.950	0.53555	0.5191	0.5006	0.4796	0.45552
0.925	0.46655	0.4488	0.4289	0.4075	0.38104
0.900	0.41495	0.3965	0.3760	0.3531	0.32712
0.85	0.33966	0.3209	0.3003	0.2772	0.2516
0.80	0.28526	0.2668	0.2468	0.2246	0.20015
0.7	0.20867	0.1919	0.1738	0.1542	0.13286
0.6	0.15541	0.1408	0.1252	0.1085	0.09094
0.5	0.11521	0.1029	0.0900	0.0764	0.06192
0.4	0.08336	0.0735	0.0633	0.0526	0.04147
0.3	0.05721	0.0498	0.0423	0.0344	0.02638
0.2	0.03524	0.0304	0.0254	0.0203	0.01513
0.1	0.01639	0.0140	0.0115	0.0106	0.00649
0	0.00000	0.00000	0.00000	0.00000	0.00000

Part. II. Matrices of Refractive Index Exceeding Unity

VI. **Introduction.** — The refractive index of the matrix of a diffuser affects its reflecting properties because not all the radiation incident on the surface from within passes through, but some is internally reflected. Hence for a given albedo for single scattering and a given phase function the reflectance still depends on the refractive index and on the state of the surface. The angular distribution of light after internal reflection at the surface has, of course, an infinite range of possibilities when all types of surface are considered, but fortunately in many cases of interest the surface may be regarded as a specular reflector, a case discussed here in some detail. An easier case to discuss is that of a hypothetical surface which diffuses perfectly all radiation reflected from or transmitted through it; its importance is that it provides a limiting case, and that the total reflectances of all real media must lie in the range established by these two cases.

Results for media with specularly reflecting surfaces and phase functions of the form $\varpi_0(1 + x \cos \theta)$ depend on the solution of integral equations, and the labour involved becomes exceedingly great as the desired degree of approximation increases. An approximate formula can be derived, however, for the reflectance, using Eddington's approximation, and this may be used to interpolate with adequate accuracy between various values of x, thus reducing considerably the effort in securing results.

Throughout Part II we use the symbol $\mathfrak{R}$ to denote the reflectance in air of a medium whose matrix may have a refractive index exceeding unity. The reflectance of a medium having the same values of ϖ_0 and x but a matrix of refractive index unity is denoted, as before, by R. With this notation, $\mathfrak{R}$ and R are identical if N (matrix) $= 1$.

VII. **Reflection and transmission at a surface of refractive index** N. — We shall first review relations between internal and external transmission at a surface of refractive index N.

Consider a constant temperature enclosure containing regions of refractive index unity and N. Distinguish quantities in the two regions by (m) and (μ) respectively, where m and μ are cosines of the angles between a light ray and the normal to the surface at the point of incidence measured in the media of refractive index unity and N respectively; then the radiation transmitted from the first to the second region in solid angle $d\Omega(m)$ is $I(m)\ mt(m)\ d\Omega(m)$, where I is the intensity and t is the transmittance of the surface; and this must equal the radiation transmitted in the reverse direction, so

$$(7.1) \qquad I(m)\ mt(m)\ d\Omega(m) = I(\mu)\ \mu t(\mu)\ d\Omega(\mu).$$

But

$$m\ d\Omega(m) = N^2\ \mu\ d\Omega(\mu),$$

and in a constant temperature enclosure

$$I(m) = \frac{I(\mu)}{N^2}.$$

Hence it follows that

$$(7.2) \qquad t(m) = t(\mu).$$

Now suppose that uniformly diffused radiation of intensity I be incident on the surface from within a medium of refractive index N, the transmitted flux is

$$\pi F = 2\pi I \int_0^1 \mu\, t(\mu)\, d\mu.$$

For rays incident at angles exceeding the critical

angle, $t(\mu)$ is zero. At other angles (7.2) applies. Thus

$$\pi F = \frac{2\pi I}{N^2} \int_0^1 m\, t(m)\, \mathrm{d}m.$$

But $2 \int_0^1 m\, t(m)\, \mathrm{d}m$ is the transmittance of the surface for diffused radiation incident from the region of refractive index unity, denoted by $\mathfrak{J}$. Then

$$(7.3) \qquad \pi F = \pi I \frac{\mathfrak{J}}{N^2}$$

so that the transmittance of the surface for diffused radiation incident from within is $\mathfrak{J}/N^2$.

The value of $\mathfrak{J}$ for an optically plane surface has been given by WALSH [6], from whose expression the following values have been computed :

N	$\mathfrak{J}$
1.333	0.933594
1.50	0.908222
1.52	0.905293

VIII. **Uniformly diffusing surfaces.** — Let flux πF_i be incident from air from any or all directions onto unit area of the surface of a medium whose matrix is of refractive index N. The surface is supposed to possess the hypothetical properties that the radiations reflected from and transmitted through it are both perfectly diffused, and the transmission and reflection properties of the surface are independent of the direction of incidence. Then the flux transmitted through the surface is $\pi F_i \mathfrak{J}$, the flux reflected at the surface is $\pi F_i(1 - \mathfrak{J})$.

Let the outward flux just below the surface of the diffuser be $\pi \mathfrak{F}$. Then of this $\pi \mathfrak{F}\mathfrak{J}/N^2$ is transmitted, and $\pi \mathfrak{F}(1 - \mathfrak{J}/N^2)$ reflected back into the medium. Thus the total inward flux immediately within the surface is $\pi F_i \mathfrak{J} + \pi \mathfrak{F}(1 - \mathfrak{J}/N^2)$, and if R_D be the total reflectance of the medium if in contact with a transparent medium of the same refractive index, then

$$R_D = \frac{\pi \mathfrak{F}}{\pi F_i \mathfrak{J} + \pi \mathfrak{F}(1 - \mathfrak{J}/N^2)},$$

whence

$$(8.1) \qquad \pi \mathfrak{F} = \frac{\pi F_i \mathfrak{J} R_D}{1 - R_D(1 - \mathfrak{J}/N^2)}.$$

Thus the reflectance $\mathfrak{R}$ of the diffuser in air, which is the ratio of the reflected flux to the incident flux, is

$$(8.2) \qquad \mathfrak{R} = \frac{\pi \mathfrak{F} \mathfrak{J}/N^2 + \pi F_i(1 - \mathfrak{J})}{\pi \mathfrak{F}_i}$$

$$= 1 - \mathfrak{J} + \frac{\mathfrak{J} R_D}{N^2 \{1 - R_D(1 - \mathfrak{J}/N^2)\}}.$$

Expressions for R_D appropriate to various phase functions have been given in Part I.

IX. **Specularly reflecting surfaces.** — Let a collimated beam of radiation of flux $\pi F_i(m)$ per unit area normal to the beam be incident on the specularly reflecting surface of a semi-infinite diffuser at an angle to the outward normal whose cosine is $-m$. Then the incident illumination is $\pi F_i(m)m$. The direct flux immediately below the surface, measured on a plane normal to the beam, is

$$\pi F_i(m) \frac{m}{\mu_0} t(m)$$

per unit area and proceeds in a direction μ_0.

As in the previous section, consider the outward radiation field just below the surface. The flux incident on unit area of the surface from the range of directions $\mathrm{d}\mu$ may be written $\pi \mathfrak{F}(\mu)\, \mathrm{d}\mu$, or $2\pi I(\mu)\, \mu\, \mathrm{d}\mu$, where $I(\mu)$ is the average value of the intensity taken around all orientations at fixed μ. This flux is reflected internally with reflection coefficient $r(\mu)$ and, together with that part of the externally incident flux which has passed through the surface, constitutes the inward radiation field just below the surface.

The outward flux results from the diffuse reflection within the medium of the two components of the inward radiation field. Denote the directional reflectance of the diffuser, in contact with a transparent medium of the same refractive index as the matrix, by $R(\mu, \mu_0)$ where μ and μ_0 are the directions of observation and incidence respectively. $R(\mu, \mu_0)$ follows immediately from expressions for the reflected intensity $I(\mu)$ in direction μ by the relation

$$(9.1) \qquad R(\mu, \mu_0) = \frac{I_r(\mu)}{F_i(\mu_0)\, \mu_0},$$

where $\pi F_i(\mu_0)\mu_0$ is the incident flux in direction μ_0 per unit area of surface. The outward flux is then given by

$$\pi \mathfrak{F}(\mu)\, \mathrm{d}\mu = 2\pi I_1\, \mu\, \mathrm{d}\mu + 2\pi I_2(\mu)\, \mathrm{d}\mu$$

where I_1 and I_2 are the intensities due to the two reflected components, i.e.

$$(9.2) \qquad \pi \mathfrak{F}(\mu)\, \mathrm{d}\mu = 2\pi F_i(m)\, m\, t(m)\, R(\mu, \mu_0)\, \mu\, \mathrm{d}\upsilon$$
$$+ 2\pi\mu\, \mathrm{d}\mu \int_0^1 \mathfrak{F}(\mu_0)\, r(\mu_0)\, R(\mu, \mu_0)\, \mathrm{d}\mu_0.$$

For some purposes it is more convenient to express (9.2) in terms of the intensity $I(\mu) = (I_1 + I_2)$ rather than the flux :

$$(9.3) \qquad I(\mu) = F_i(m)\, m\, t(m)\, R(\mu, \mu_0) +$$
$$+ 2 \int_0^1 I(\mu_0)\, r(\mu_0)\, R(\mu, \mu_0)\, \mu_0\, \mathrm{d}\mu_0.$$

The flux transmitted outwards through the surface is

$$(9.4) \qquad \pi F_r = 2\pi \int_0^1 I(\mu)\, \mu\, t(\mu)\, \mathrm{d}\mu$$

and the total reflectance $\mathfrak{R}$ is then obtained from the relation

$$(9.5) \qquad \mathfrak{R} = \frac{\pi F_r}{\pi F_i(m)\, m} + r$$

where the first term on the right hand side represents the diffuse reflectance and the second term the specular reflectance.

X. **Scattering according to a phase function $\varpi_0(1 + x \cos\theta)$; Eddington's approximation.** — Eddington's approximation has the merit that while it yields results of only limited accuracy, it does enable a general expression to be obtained for the reflectance, incorporating the albedo for single scattering and the parameter x in the phase function $\varpi_0(1 + x\cos\theta)$.

By multiplying (9.3) by μ and integrating with respect to μ, the equation is found

$$(10.1) \qquad \int_0^1 I(\mu)\,\mu\, d\mu = \frac{1}{2} F_i(m)\, m\, t(m)\, R(\mu_0) + \int_0^1 I(\mu_0)\, r(\mu_0)\, R(\mu_0)\, \mu_0\, d\mu_0,$$

where

$$R(\mu_0) = 2\int_0^1 \mu\, R(\mu, \mu_0)\, d\mu$$

is the total reflectance for radiation incident in direction μ_0 on the diffuser when in contact with a transparent medium of the same refractive index as the matrix.

An approximate solution of (10.1) is obtained by assuming $I(\mu)$ to be constant with μ. Then

$$I = F_i(m)\, m\, t(m)\, R(\mu_0) + 2 I \mathfrak{J},$$

where

$$\mathfrak{J} = \int_0^1 r(\mu_0)\, R(\mu_0)\, \mu_0\, d\mu_0$$

so that

$$(10.2) \qquad I = \frac{F_i(m)\, m\, t(m)\, R(\mu_0)}{1 - 2\mathfrak{J}}\,.$$

Since with the Eddington approximation $R(\mu_0)$ is given by (4.1), equation (10.2) provides an analytic expression for I in terms of the phase function $\varpi_0(1 + x\cos\theta)$.

From (7.3), the flux transmitted outwards through the surface when $I(\mu)$ is constant with μ is

$$\pi F_r = \frac{\pi I \mathfrak{J}}{N^2}$$

so that from (9.5) the total reflectance of the diffuser is

$$(10.3) \qquad \mathfrak{R} = \frac{t(m)\, R(\mu_0)\, \mathfrak{J}}{N^2(1 - 2\mathfrak{J})} + r.$$

XI. **Solutions of higher accuracy.** — The solution obtained above on the assumption of constant $I(\mu)$ lies only part way towards the exact solution. To obtain improved accuracy, account must be taken of the variation of $I(\mu)$ with μ, and this may be done by replacing the integral in (9.3) by a quadrature formula. The most appropriate division of the interval of μ_0 from 0 to 1 offers considerable difficulty, for ideally a Gaussian division should be made. However, such a division depends on the weighting function $r(\mu_0)\, R(\mu, \mu_0)\, \mu_0$, and while $r(\mu_0)$ is small provided $\mu_0 > \mu_c$, where μ_c is the cosine of the critical angle, it is just this part of the division which is important when (9.4) is to be evaluated subsequently. Further, $R(\mu, \mu_0)$ depends on both ϖ_0 and μ, so that different divisions would be best for every ϖ_0 and every μ, which is clearly impracticable. Fortunately, the range of variation of $R(\mu, \mu_0)$ with μ_0 is not very great when μ_0 is not too small. The interval will therefore be divided with the simple weighting function μ_0, in which case $\int_0^1 I(\mu_0)\, r(\mu_0)\, R(\mu, \mu_0)\, \mu_0\, d\mu_0$ in (9.3) may be replaced, for greatest convenience in subsequent computation, by $\sum_j I(\mu_j)\, R(\mu_k, \mu_j)\, a_j$ where according to the method of Gaussian division, the weight a_j attached to the division μ_j is

$$(11.1) \qquad a_j = \frac{1}{\prod_{i \neq j} (\mu_j - \mu_i)} \int_0^1 \left[\prod_{i\neq j} (\mu - \mu_i)\right] \mu\, r(\mu)\, d\mu\,.$$

Furthermore, the division points μ_j are such that

$$(11.2) \qquad \int_0^1 f(\mu)\,\mu\, d\mu = \sum_{j=1}^{n} c_j\, f(\mu_j)$$

where $f(\mu)$ is any polynomial of degree $2n - 1$ and the c_j are appropriate numerical constants. Let

$$f(\mu) = b_0 + b_1\mu + b_2\mu^2 + \ldots\ldots\ldots$$

Then on substituting in (11.2) and equating coefficients of b_j, there follows the set of equations whose solutions give μ_j :

$$(11.3) \qquad \sum_{j=1}^{n} c_j\, \mu_j^r = \frac{1}{r+2}$$

where in successive equations r takes the values from 0 to 2n-1. Values of the division μ_j for small values of n are

n	μ_j
1	$\mu_1 = 0.667$
2	$\mu_1 = 0.356$ $\mu_2 = 0.845$
3	$\mu_1 = 0.23$ $\mu_2 = 0.667$ $\mu_3 = 0.96$

On making the above substitutions, (9.3) is replaced by the set of n simultaneous equations

$$(11.4) \quad I(\mu_k) = F_i(m)\, m\, t(m)\, R(\mu_k, \mu_o) + 2 \sum_{j=1}^{n} I(\mu_j)\, R(\mu_k, \mu_j)\, a_j,$$

where in successive equations k takes the values from 1 to n. The solution of the set of equations (11.4) yields $I(\mu_k)$ to the nth approximation.

With the values of $I(\mu_k)$ solved, the flux passing out into the medium of incidence is given by (9.4), which is in turn given to the nth approximation by

$$(11.5) \qquad \pi F_r = 2\pi \sum_{j=1}^{n} A_j\, I(\mu_j)$$

where the weight A_j attached to the division μ_j is now

$$(11.6 \quad A_j = \frac{1}{\prod_{i \neq j} (\mu_j - \mu_i)} \int_0^1 \left[\prod_{i \neq j} (\mu_o - \mu_i) \right] \mu_0\, t(\mu_0)\, d\mu_0.$$

The total reflectance follows from an application of (9.5).

Since the effort involved increases greatly with the order of the approximation, it is desirable to know the accuracy achieved with various values of n. The first three approximations do show appreciable differences, as the following examples indicate.

Isotropic diffuser, specularly reflecting surface, N (matrix) = 1.5 ; $\mu_0 = 1$

ϖ_0	1st approx.	2nd approx.	3rd approx.
1.000	1.0151	1.0228	1.00019
0.975	0.4671	0.4527	0.4685

The exact result for $\varpi_0 = 1$ is 1.00000, so that the third approximation is in error by only 2 parts in ten thousand. This does not automatically mean a higher approximation would necessarily give a better result, or that the third approximation is necessarily satisfactory for other values of ϖ_0, though it seems likely that the accuracy will improve as the absorption increases.

Two further checks have been obtained by carrying out the computation of the reflectance on the third approximation for (i) a polished medium of refractive index 1.333 and $\varpi_0 = 1$, and (ii) the same medium in contact with a cover glass of refractive index 1.523. The results are 1.00011 and 1.00012 as compared with the exact values 1.00000.

Finally, the third approximation has been used for the case $N = 1$; the results may be compared with the exact values given in Table I, and for a few values of ϖ_0 are.

ϖ_0	R (third approx.)	R (exact)
1.000	0.99999	1.00000
0.975	0.64093	0.64092
0.800	0.28526	0.28526

The agreement is excellent, whereas the first approximation gives an error of about 1 per cent in R.

There are therefore good grounds for believing that, for $N > 1$ also, the third approximation will yield results of adequate accuracy for all photometric purposes.

XII. **Results.** — Table 4 gives the total reflectance of a diffuser scattering isotropically, of refractive index 1.5, with a perfectly diffusing surface (equation 8.2) ; the values of R_D have been taken from Table 1.

Table 4

Total reflectance for a diffuser with a perfectly diffusing surface. Isotropic scattering. N (matrix) = 1.5

ϖ_0	Reflectance	ϖ_0	Reflectance
1.000	1.00000	0.80	0.2492
0.999	0.8567	0.7	0.2028
0.995	0.7235	0.6	0.1725
0.990	0.6454	0.5	0.1506
0.975	0.5269	0.4	0.1347
0.950	0.4314	0.3	0.1203
0.925	0.3756	0.2	0.1092
0.900	0.3369	0.1	0.0998
0.85	0.2845	0	0.09178

In Table 5 results are given for the total reflectances of diffusers scattering isotropically or according to a phase function $\varpi_0(1 + \cos\theta)$, as derived on the third approximation. The surface is specularly reflecting and the matrix of refractive index 1.5. Values of $d\mathfrak{R}/dN$, obtained by computing $\mathfrak{R}$ on the first approximation for matrices of refractive index 1.50 and 1.52, are also included, these enabling the results to be extended over a useful range of refractive index. As

TABLE 5

Total reflectances for isotropic scattering and for a phase function $\varpi_0(1 + \cos\theta)$, obtained on the third approximation. Gradients $d\mathfrak{R}/dN$ derived on the first approximation. N (matrix) = 1.5 ; specularly reflecting surface ; $\mu_0 = 1$.

ϖ_0	Isotropic		$\varpi_0(1 + \cos\theta)$	
	$\mathfrak{R}$	$d\mathfrak{R}/dN$	$\mathfrak{R}$	$d\mathfrak{R}/dN$
1.000	1.00000	0.0000	1.00000	0.0000
0.999	0.8414	—0.162	0.8107	—0.186
0.995	0.6910	—0.251	0.6392	—0.263
0.990	0.6020	—0.271	0.5418	—0.270
0.975	0.4685	—0.260	0.4003	—0.233
0.950	0.3612	—0.216	0.2936	—0.170
0.925	0.3008	—0.177	0.2355	—0.123
0.900	0.2600	—0.145	0.1975	—0.087
0.8	0.1720	—0.058	0.1206	0.000
0.7	0.1289	—0.006	0.0868	+0.045
0.6	0.1024	+0.030	0.0683	+0.073
0.5	0.0841	+0.057	0.0571	+0.091
0.4	0.0708	+0.077	0.0500	+0.104
0.3	0.0605	+0.094	0.0455	+0.114
0.2	0.0523	+0.107	0.0426	+0.120
0.1	0.0456	+0.119	0.0409	+0.125
0	0.04000	+0.129	0.04000	+0.129

indicated in § XI, there are grounds for believing that the errors in the reflectances derived on the third approximation are confined to the fourth decimal.

TABLE 6

Reflectances obtained with the EDDINGTON approximation for phase functions of the forms $\varpi_0(1 + x\cos\theta)$. N (matrix) = 1.5, specularly reflecting surface ; $\mu_0 = 1$.

ϖ_0	Isotropic		$\varpi_0(1 + \cos\theta)$		$\varpi_0(1 + 0.5\cos\theta)$	
	$\mathfrak{R}$	Correction	$\mathfrak{R}$	Correction	$\mathfrak{R}$	Corrected $\mathfrak{R}$
1.000	1.0000	0.0000	1.0000	0.0000	1.0000	1.0000
0.999	0.8426	—0.0012	0.8137	—0.0030	0.8311	0.8290
0.995	0.6959	—0.0049	0.6483	—0.0091	0.6748	0.6678
0.990	0.6101	—0.0081	0.5532	—0.0114	0.5850	0.5752
0.975	0.4808	—0.0123	0.4166	—0.0163	0.4520	0.4377
0.950	0.3780	—0.0168	0.3130	—0.0194	0.3486	0.3305
0.925	0.3189	—0.0181	0.2561	—0.0206	0.2904	0.2710
0.900	0.2785	—0.0185	0.2185	—0.0210	0.2511	0.2313
0.8	0.1890	—0.0170	0.1403	—0.0197	0.1663	0.1479
0.7	0.1431	—0.0142	0.1040	—0.0172	0.1247	0.1090
0.6	0.1141	—0.0117	0.0828	—0.0145	0.0992	0.0861
0.5	0.0934	—0.0093	0.0690	—0.0119	0.0817	0.0711
0.4	0.0786	—0.0078	0.0594	—0.0094	0.0689	0.0603
0.3	0.0655	—0.0050	0.0524	—0.0069	0.0591	0.0531
0.2	0.0555	—0.0032	0.0472	—0.0046	0.0514	0.0475
0.1	0.0471	—0.0015	0.0431	—0.0022	0.0451	0.0433
0	0.0400	0.0000	0.0400	0.0000	0.0400	0.0400

Comparisons are made in Table 6 between the results of Table 5 and those obtained for the same diffusers on the EDDINGTON approximation. The corrections required to results obtained on the EDDINGTON approximation to yield those obtained on the more accurate third approximation are almost the same for $x = 0$ and $x = 1$, so that the simpler EDDINGTON approximation may be used for intermediate cases of the phase function $\varpi_0(1 + x\cos\theta)$ and the appropriate correction found by linear interpolation. Results obtained in this way for $x = 0.5$ (column 7) should be in error by less than 1 in the third decimal place, and this has been confirmed by comparison with a few results obtained on the third approximation. From the data of Tables 5 and 6, reflectances for other phase functions of the type $\varpi_0(1 + x\cos\theta)$ may readily be obtained using three-point interpolation between the values of x. Further, it may be noted that to within $\pm$ 0.001 the reflectance for $x = 0.5$ divides the interval between reflectances for $x = 0$ and 1, for the same ϖ_0, in the ratio 0.45 for both $N = 1$ and 1.5, and the same division is therefore applicable for intermediate values of N.

TABLE 7

Total reflectances for isotropic scattering and for a phase function $\varpi_0(1 + \cos\theta)$, obtained with the third approximation and with the EDDINGTON approximation. N (matrix) = 1.333, polished surface, $\mu_0 = 1$.

ϖ_0	Third approximation		EDDINGTON approximation		Correction required to EDDINGTON approximation	
	Isotropic	$\varpi_0(1 + \cos\theta)$	Isotropic	$\varpi_0(1 + \cos\theta)$	Isotropic	$\varpi_0(1 + \cos\theta)$
1.000	1.000	1.0000	1.0000	1.0000	0.0000	0.0000
0.99	0.6520	0.5934	0.6584	0.6031	—0.0064	—0.0097
0.95	0.4073	0.3324	0.4229	0.3512	—0.0156	—0.0188
0.80	0.1898	0.1271	0.2082	0.1486	—0.0184	—0.0215
0.60	0.1030	0.0596	0.1161	0.0768	—0.0131	—0.0172
0	0.0204	0.0204	0.0204	0.0204	0.0000	0.0000

Table 7 contains reflectances computed on the third approximation for specularly reflecting media having N(matrix) = 1.333, and phase functions $\varpi_0(1 + x\cos\theta)$, where $x = 0$ or 1 ; these are compared with results from the EDDINGTON approximation, and it will be seen that the corrections required in the latter case are very nearly the same as those of Table 6. Corrections for intermediate values of N can readily be interpolated, and even without interpolation, cor-

TABLE 8

Total reflectances for isotropic scattering and for a phase function $\varpi_0(1 + \cos\theta)$, obtained on the third approximation. N (matrix) = 1.333, in contact with a cover glass of refractive index 1.523 ; $\mu_0 = 1$.

ϖ_0	Reflectance	
	Isotropic	$\varpi_0(1 + \cos\theta)$
1.000	1.00000	1.00000
0.999	0.8684	0.8420
0.995	0.7361	0.6886
0.990	0.6541	0.5966
0.975	0.5242	0.4559
0.950	0.4155	0.3436
0.925	0.3512	0.2799
0.900	0.3067	0.2372
0.8	0.2072	0.1476
0.7	0.1565	0.1068
0.6	0.1246	0.0838
0.5	0.1021	0.0697
0.4	0.0858	0.0607
0.3	0.0729	0.0547
0.2	0.0626	0.0509
0.1	0.0541	0.0484
0	0.04702	0.04702

rections from Table 6 can be used to an accuracy normally sufficient for any N in the range $1.333 < N < 1.5$. Table 8 contains reflectances obtained on the third approximation for a medium scattering isotropically or with a phase function $\varpi_0(1 + \cos\theta)$, the matrix being of refractive index 1.333 and the medium in optical contact with a cover glass of refractive index 1.523. The results in Table 8 apply specifically to diffusing media whose matrix is water, contained in a glass cell ; the refractive index of the latter is usually close to 1.523. Comparison with Table 7 shows that the cover glass has little effect for total reflectances above about 0.6.

TABLE 9

Directional reflectances for light incident along the normal. N (matrix) = 1.5 ; specularly reflecting surface.
Isotropic scattering

ϖ_0	Reflectance					
	angle to normal					
	0°	15°	30°	45°	60°	75°
1.0000	1.0164	1.016	1.014	1.0040	0.962	0.788
0.999	0.8440	0.844	0.844	0.8381	0.806	0.661
0.995	0.6814	0.682	0.683	0.6807	0.657	0.540
0.990	0.5858	0.587	0.589	0.5870	0.568	0.468
0.975	0.4437	0.445	0.447	0.4483	0.435	0.360
0.950	0.3302	0.331	0.334	0.3360	0.328	0.272
0.925	0.2670	0.267	0.270	0.2728	0.267	0.221
0.900	0.2245	0.225	0.228	0.2301	0.225	0.188
0.85	0.1691	0.169	0.172	0.1742	0.171	0.143
0.80	0.1336	0.134	0.136	0.1371	0.136	0.114
0.7	0.0896	0.090	0.091	0.0930	0.092	0.077
0.6	0.0627	0.063	0.064	0.0653	0.064	0.054
0.5	0.0443	0.044	0.045	0.0463	0.045	0.038
0.4	0.0309	0.031	0.032	0.0323	0.032	0.027
0.3	0.0205	0.021	0.021	0.0215	0.021	0.018
0.2	0.0123	0.012	0.013	0.0129	0.013	0.011
0.1	0.0056	0.006	0.006	0.0059	0.006	0.005

TABLE 10

Directional reflectances for light incident along the normal. N (matrix) = 1.5 ; specularly reflecting surface.
Phase function : $\varpi_0(1 + \cos\theta)$

ϖ_0	Reflectance					
	angle to normal					
	0°	15°	30°	45°	60°	75°
1.0000	1.0164	1.016	1.014	1.0040	0.962	0.788
0.999	0.8104	0.811	0.811	0.806	0.776	0.637
0.995	0.6248	0.626	0.628	0.627	0.606	0.499
0.990	0.5201	0.521	0.524	0.525	0.510	0.421
0.975	0.3688	0.370	0.374	0.377	0.368	0.306
0.950	0.2559	0.257	0.261	0.265	0.262	0.219
0.925	0.1950	0.196	0.200	0.204	0.202	0.170
0.900	0.1555	0.157	0.160	0.164	0.164	0.139
0.8	0.0765	0.078	0.081	0.084	0.086	0.073
0.7	0.0426	0.043	0.046	0.049	0.051	0.044
0.6	0.0245	0.025	0.027	0.029	0.032	0.028
0.5	0.0139	0.014	0.016	0.018	0.019	0.018
0.4	0.0075	0.008	0.009	0.010	0.012	0.012
0.3	0.0036	0.004	0.005	0.006	0.007	0.006
0.2	0.0014	0.002	0.002	0.003	0.003	0.003
0.1	0.0003	0.001	0.001	0.001	0.001	0.001

TABLE 11

Directional reflectances for light incident along the normal. N (matrix) = 1.333, medium in contact with a cover glass, N = 1.523
Isotropic scattering

ϖ_0	Reflectance					
	angle to normal					
	0°	15°	30°	45°	60°	75°
1.000	1.012	1.011	1.008	1.001	0.953	0.829
0.999	0.865	0.866	0.866	0.863	0.825	0.720
0.995	0.721	0.722	0.724	0.724	0.695	0.609
0.990	0.631	0.633	0.635	0.637	0.614	0.540
0.975	0.491	0.493	0.496	0.501	0.485	0.428
0.950	0.377	0.378	0.382	0.387	0.377	0.334
0.925	0.308	0.310	0.314	0.320	0.312	0.278
0.900	0.262	0.263	0.267	0.273	0.267	0.238
0.8	0.160	0.161	0.164	0.168	0.166	0.149
0.7	0.109	0.110	0.111	0.114	0.113	0.102
0.6	0.077	0.078	0.079	0.081	0.081	0.073
0.5	0.054	0.054	0.056	0.057	0.057	0.052
0.4	0,038	0.038	0.039	0.041	0.040	0.036
0.3	0.025	0.026	0.027	0.027	0.027	0.025
0.2	0.015	0.015	0.016	0.016	0.016	0.015
0.1	0.008	0.008	0.008	0.008	0.008	0.007

TABLE 12

Directional reflectances for light incident along the normal. N (matrix) = 1.333, medium in contact with a cover glass, N = 1.523.
Phase function : $\varpi_0(1 + \cos\theta)$

ϖ_0	Reflectance					
	angle to normal					
	0°	15°	30°	45°	60°	75°
1.000	1.012	1.011	1.008	1.001	0.953	0.829
0.999	0.836	0.837	0.837	0.835	0.799	0.698
0.995	0.668	0.669	0.672	0.674	0.649	0.570
0.990	0.567	0.568	0.572	0.577	0.558	0.492
0.975	0.415	0.417	0.423	0.430	0.419	0.373
0.950	0.295	0.297	0.304	0.311	0.307	0.275
0.925	0.229	0.231	0.237	0.245	0.243	0.219
0.900	0.183	0.186	0.191	0.200	0.201	0.182
0.8	0.092	0.094	0.098	0.105	0.108	0.101
0.7	0.052	0.053	0.057	0.062	0.067	0.063
0.6	0.030	0.031	0.034	0.039	0.043	0.041
0.5	0,017	0.018	0.020	0.024	0.027	0.026
0.4	0.009	0.010	0.011	0.015	0.017	0.017
0.3	0.004	0.005	0.006	0.008	0.010	0.011

In all the above results the total reflectances include the specular component, whose magnitude is the reflectance for $\varpi_0 = 0$. Diffuse reflectances are readily obtained by subtraction. Further, the values for $\varpi_0 = 1$ are the exact values, rather than the slightly different values obtained on the third approximation.

Directional reflectances for normally incident radiation, as derived from the third approximation, for various media with specularly reflecting surfaces are given in Tables 9-12. In a few cases results are quoted to four decimal places, and these have been computed

using (11.4). Results given to three decimals have been obtained using graphical interpolation between the $I(\mu_j)$; comparison with the computed values has shown that the results from graphical interpolation should be reliable to one unit in the third decimal place. In obtaining reflectances, account has been taken of the transmittance of the surface both for the incident and the reflected radiation. For the directional reflectance along the normal, the specular component has not been included. A few selected computations for these cases have indicated that to within $+ 0.001$ the directional reflectance for $x = 0.5$ divides the interval between those for $x = 0$ and 1, for the same ϖ_0, in the ratio 0.45.

XIII. **Acknowledgment.** — The author wishes to thank Miss J. WARD for assistance with the computations.

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BIBLIOGRAPHIE

Recent advances in Optics, par E. H. LINFOOT, un volume relié toile de x-286 pages (23,5 × 15 cm), illustré de 113 figures et 8 planches hors-texte, Oxford University Press, 1955 (*50 s*).

Cet ouvrage traite essentiellement des progrès effectués récemment dans la théorie de la formation des images optiques et dans la conception des instruments astronomiques. Il comprend quatre parties.

Dans la première « The optical image », l'auteur étudie d'abord l'aspect géométrique de la formation des images.

Après avoir donné (dans un formalisme nouveau où l'on utilise des quantités complexes) une classification des termes d'aberrations, l'auteur s'attache à rechercher une méthode de réduction optimum des aberrations, basée sur la réduction du rayon de giration de l'image géométrique.

On passe ensuite à l'étude de la tache de diffraction en présence d'aberrations et une documentation assez abondante est fournie sur les résultats obtenus par divers auteurs (cas d'aberration sphérique, de coma, d'astigmatisme ou de superposition). On aborde ensuite le problème de la cohérence partielle par une méthode très voisine de celle de HOPKINS et les relations de base sont établies. Le contraste de phase est enfin considéré comme un cas particulier.

Dans la seconde partie « The Foucault test », l'auteur présente une théorie détaillée de la méthode de FOUCAULT : basée sur des considérations géométriques, cette méthode met manifestement en jeu des phénomènes de diffraction. L'auteur justifie en optique ondulatoire les indications de l'optique géométrique, en mettant néanmoins en évidence l'influence de la diffraction (par exemple, la présence de franges brillantes au voisinage de discontinuités et aussi en discutant les erreurs systématiques qui peuvent résulter de l'interprétation purement géométrique de figures de Foucault en présence de faibles aberrations zonales).

La troisième partie « The Schmidt camera » est consacrée à la chambre de SCHMIDT et à ses dérivées. On étudie par la méthode des « spots diagrams » les aberrations géométriques de l'objectif original et l'on propose diverses solutions pour équilibrer les aberrations sur la surface image sphérique et pour obtenir un champ plan.

Enfin, dans la quatrième partie « Plate diagramm analysis and its applications », l'auteur étudie la méthode d'analyse des aberrations du troisième ordre (due à BURCH) où les aberrations de chaque dioptre sont représentées par une lame déformée fictive située dans le plan du centre ; il l'applique ensuite au cas de l'objectif de CASSEGRAIN et de SCHMIDT-CASSEGRAIN (proposé par BAKER et par l'auteur).

L'ensemble apporte une riche documentation et l'auteur nous fait pleinement profiter de son expérience sur la résolution mathématique de nombreux problèmes fondamentaux d'optique instrumentale. L'ouvrage est abondamment illustré (en particulier de documents relatifs aux taches de diffraction en présence d'aberrations) ce qui en facilite beaucoup la lecture (A. MARÉCHAL).

Radiative transfer in two dimensions

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SUMMARY. — *A first approximation is obtained for the distribution of intensity in a semi infinite scattering medium illuminated by normally incident radiation whose flux is an arbitrary function of one Cartesian coordinate in the boundary plane.*

SOMMAIRE. — *Une première approximation est obtenue pour la distribution de l'intensité dans un milieu diffusant éclairé sous l'incidence normale par un rayonnement dont le flux est une fonction arbitraire de l'une des coordonnées cartésiennes dans le plan frontière.*

ZUSAMMENFASSUNG. — *Es wird eine erste Näherung für die Intensitätsverteilung in dem Halbraum eines streuenden Mediums unter folgenden Annahmen aufgestellt : Die Strahlung fällt senkrecht ein, wobei der Lichtstrom eine willkürliche Funktion einer Koordinate der Grenzfläche ist.*

1. **Introduction**. — The diffusion of radiation through a scattering medium is described by the equation of transfer whose solution gives the specific intensity I in the medium as a function of position and direction. Attention, so far, has been mainly confined to plane parallel media for which the sources of radiation and the physical properties are uniform over any plane parallel to the bounding surface. In these cases, clearly, the space variation of I can be described in terms of a single coordinate, namely the depth in the medium.

Among the problems of this one dimensional type which have been solved, is that of the intensity distribution in a plane semi-infinite scattering medium uniformly illuminated by normally incident radiation. The main purpose of this paper is to derive a first approximation to the solution of the transfer equation in the corresponding two dimensional case, i. e. when the normally incident flux has arbitrary variation along one Cartesian axis in the boundary plane and is constant along the other.

Two dimensional transfer problems occur naturally in many connexions, an example is in the discussion of the resolving power of a photographic emulsion where sideways scattering of radiation is of prime importance.

2. **The equation of transfer**. — For an isotropically scattering medium the equation of transfer may be written (CHANDRASEKHAR [1])

$$(1) \qquad -\frac{1}{\varkappa}\frac{dI}{ds} = I - \frac{\varpi_0}{4\pi}\int I \, d\omega$$

where I is the specific intensity in the direction S, ϖ_0 is the albedo for single scattering and $\varkappa$ is the attenuation or extinction coefficient. In a Cartesian coordinate system (1) becomes

$$(2) \quad -\frac{1}{\varkappa}\left[l\frac{\partial I}{\partial x} + m\frac{\partial I}{\partial y} + n\frac{\partial I}{\partial z}\right] = I - \frac{\varpi_0}{4\pi}\int I \, d\omega$$

where l, m and n are direction cosines of I and the integral is taken over a solid angle 4π. Equation (2) is the basic equation of the problem whose solution gives I as a function of position and direction.

A number of methods for the approximate solution of (2) in the one dimensional case have been given (see e. g. KOURGANOFF [2]). In the method of discrete ordinates, which will be used here, the radiation field is divided into streams in certain specified directions and the intensity in each of these directions determined. The points of division of the radiation field having been made, the integral in (2) is replaced by a suitably weighted sum over the intensities in these directions. As the number of points of division is made greater and greater, the calculated intensity in any direction may be expected to tend to a solution of (2). If the radiation field is axially symmetric, the integration over the azimuth can be carried out at once and only an approximation to the integral over the polar angle is required. In the problems discussed here, I depends on both polar and azimuthal angles θ and φ and an approximation to the double integral over these angles is required.

In two-dimensional problems, suppose I to be independent of the y coordinate and divide the radiation field into beams with azimuths — measured from the x axis — of $\pm\pi/4$ and $\pm 3\pi/4$. The approximation to the integral over φ then becomes

$$\int_0^{2\pi} I \, d\varphi \simeq \frac{\pi}{2}\sum_{i=1}^{4} I(\varphi_i).$$

For the division of θ, the angle which the beam makes with the z axis, we find it convenient to make a division into the directions $\mu_j \equiv \cos\theta_j = \pm 1, \pm\varepsilon$ where ε is small. It will be noticed that the beams in the directions $\mu = \pm 1$ have an unspecified azimuth ; however, each may be regarded as the resultant of four degenerate beams with azimuths $\pm\pi/4$, $\pm 3\pi/4$. The purpose of the division into $\pm\varepsilon$ instead of simply

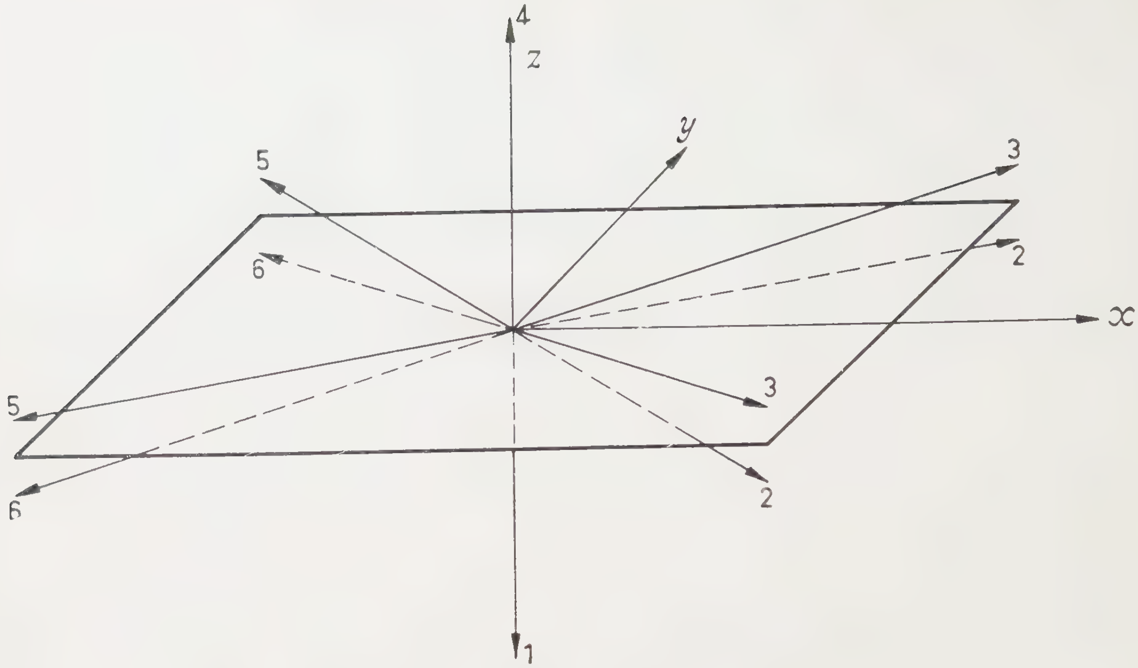

FIG. 1. — Directions of the beams into which the radiation field is divided

taking $\mu = 0$ is to enable us to formulate correct boundary conditions before taking the limit $\varepsilon \to 0$. Attempts to treat a division with any other values of μ_j have led to considerable mathematical complexities.

For any particular azimuth φ_i, it then follows that with the above division for μ,

$$\int_{-1}^{+1} I(\varphi_i, \mu)\, d\mu \simeq \sum_{j=1}^{4} b_j I(\varphi_i, \mu_j)$$

where b_j is 1/3 for $\mu_j = \pm 1$ and 2/3 for $\mu_j = \pm \varepsilon$. The approximation to the integral over the solid angle may thus be written,

$$(3) \qquad \int_0^{2\pi}\int_{-1}^{+1} I\, d\mu\, d\varphi \simeq \frac{\pi}{6}\Big[\Sigma\, I(\varphi_i, +1) + + 2\,\Sigma\, I(\varphi_i, +\varepsilon) + 2\,\Sigma\, I(\varphi_i, -\varepsilon) + \Sigma\, I(\varphi_i, -1) \ldots\Big]$$

the summations being over the four directions of φ_i. From the symmetry of the problem we have $I(\varphi_i, \mu_i) = I(-\varphi_i, \mu_i)$ so that the integral in (2) may be written

$$(4) \qquad \frac{\varpi_0}{4\pi}\int I\, d\omega \sim \frac{\varpi_0}{6}\sum_{i=1}^{6} I_i .$$

The directions of the beams are shown in figure 1 where the beams of equal intensities in the directions $\pm\,\varphi_i$ are indicated by the same number.

3. **Collimated normally incident radiation.** — Consider a plane parallel isotropically scattering medium illuminated by normally incident radiation of flux $F(x)$ per unit area. The transfer equation for the diffuse radiation in the medium may be written, according to the above approximation,

$$(5) \qquad -l_i \frac{\partial I_i}{\partial u} - n_i \frac{\partial I_i}{\partial v} = I_i - \frac{\varpi_0}{6}\Sigma\, I_i - \frac{\varpi_0}{4\pi} F(u)\, e^{-v}$$

where $u = \varkappa x$, $v = \varkappa z$ and the last term arises from scattering of the incident flux which has penetrated to a depth z. We try to solve the set (5) by a FOURIER transform method.

The FOURIER transform with respect to u of equations (5) may be written

$$(6) \qquad -n_i \frac{\partial \boldsymbol{I}_i}{\partial v} = \boldsymbol{I}_i(1 - l_i\, i\, \xi) - \frac{\varpi_0}{6}\Sigma\, \boldsymbol{I}_i - \frac{\varpi_0}{4\pi}\psi(\xi)\, e^{-v}$$

where

$$\boldsymbol{I}_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi u}\, I(u, v)\, du,$$

and

$$\psi(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi u}\, F(u)\, du\, .$$

Substituting for n_i and l_i, six first order simultaneous differential equations are obtained whose solutions may be found in the usual manner. For the homogeneous part of (6),

$$(7) \qquad \boldsymbol{I}_i = f_i\, e^{-k_1 v} + g_i\, e^{-k_2 v} + h_i\, e^{-k_3 v}$$

where the f_i, g_i and h_i are functions of the parameter ξ

alone and the k's are obtained as solutions of the cubic equation in k^2,

$$(8)\quad \frac{1}{\gamma} = \left[\frac{1}{1-k^2} + \frac{1}{1-(\eta+\varepsilon k)^2} + \frac{1}{1-(\eta-\varepsilon k)^2}\right]$$

where $\eta = i\,\xi/\sqrt{2}$ and $\gamma = \varpi_0/3$. This equation has roots, to the first order in ε,

$$(9)\quad k^2 = \frac{1-3\gamma-(1-\gamma)\eta^2}{1-2\gamma-\eta^2}\ ;$$
$$(\varepsilon k)^2 = 1-\gamma+\eta^2 \pm \sqrt{\gamma^2+4\eta^2(1-\gamma)}\ .$$

The negative roots for k are rejected as incompatible with the requirement that $I_i \to 0$ as $v \to \infty$.

The particular integral is found by assuming a solution of the form $I_i = p_i \frac{\varpi_0}{4\pi}\, e^{-v}\,\psi(\xi)$ whence it follows, on substitution in (6) that $p_i = 0$, $i \neq 1$, $p_1 = -2/\gamma$.

The integration constants f_i, g_i and h_i are found from the conditions $I_1 = I_2 = I_6 = 0$ at $v = 0$. These give

$$(10)\quad \begin{cases} f_1 + g_1 + h_1 = (3/2\pi)\,\psi(\xi) \\ f_2 + g_2 + h_2 = 0 \\ f_3 + g_3 + h_3 = 0 \end{cases}$$

which are equivalent to the equations

$$(11)\quad \begin{cases} \dfrac{s(f)}{1-k_1} + \dfrac{s(g)}{1-k_2} + \dfrac{s(h)}{1-k_3} = \dfrac{3}{2\pi}\,\psi(\xi) \\[2ex] \dfrac{s(f)}{1-\eta-\varepsilon k_1} + \dfrac{s(g)}{1-\eta-\varepsilon k_2} + \dfrac{s(h)}{1-\eta-\varepsilon k_3} = 0 \\[2ex] \dfrac{s(f)}{1+\eta-\varepsilon k_1} + \dfrac{s(g)}{1+\eta-\varepsilon k_2} + \dfrac{s(h)}{1+\eta-\varepsilon k_3} = 0 \end{cases}$$

where $s(f) = \dfrac{\varpi_0}{6}\,\Sigma f_i$ and $s(g)$ and $s(h)$ are similarly defined.

Now letting $\varepsilon \to 0$, two of the k's, say k_2 and k_3, tend to infinity in such a way that εk_2 and εk_3 remain finite, being given by (9). It follows then, from (11) that

$$(12)\quad \begin{cases} s(f) = \dfrac{3}{2\pi}\,(1-k_1)\,\psi(\xi) \\[2ex] s(g) = -\dfrac{3}{2\pi}\,\dfrac{(1-k_1)}{1-\eta^2}\left[(1-\eta-\zeta_2)(1+\eta-\zeta_2)\,\dfrac{\zeta_3}{\zeta_3-\zeta_2}\right]\psi(\xi) \\[2ex] s(h) = -\dfrac{3}{2\pi}\,\dfrac{(1-k_1)}{1-\eta^2}\left[(1-\eta-\zeta_3)(1+\eta-\zeta_3)\,\dfrac{\zeta_2}{\zeta_2-\zeta_3}\right]\psi(\xi) \end{cases}$$

where $\zeta_j = \varepsilon\, k_j$.

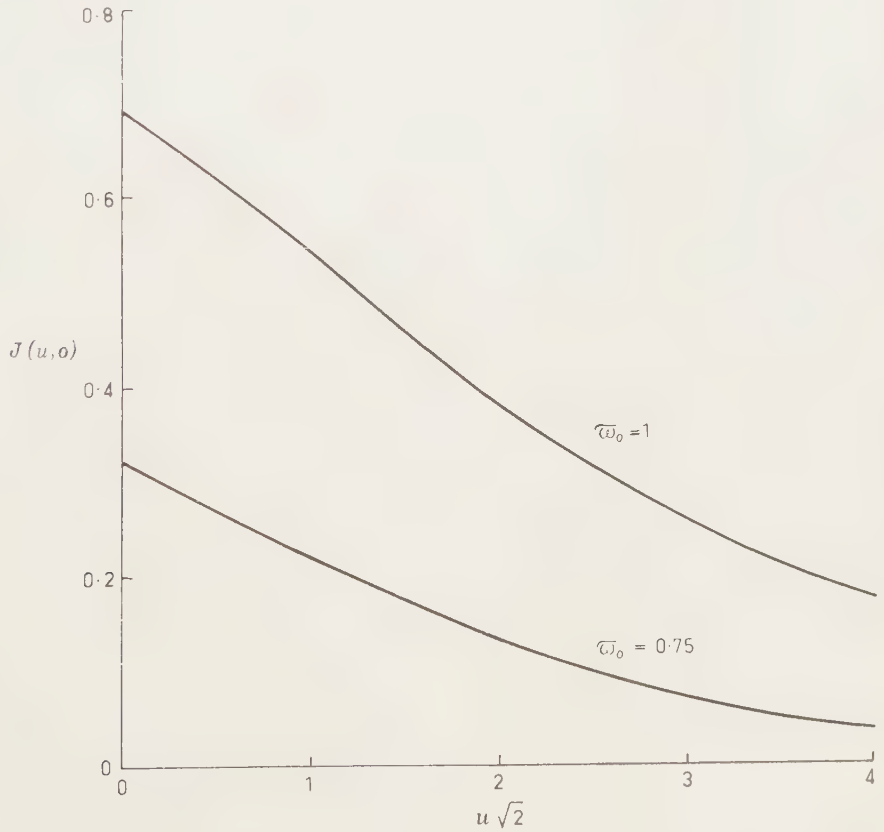

FIG. 2. — The variation of intensity at the surface for $\varpi_0 = 1$ and 0.75 with an incident flux $F(u) = e^{-|u|\sqrt{2}}$.

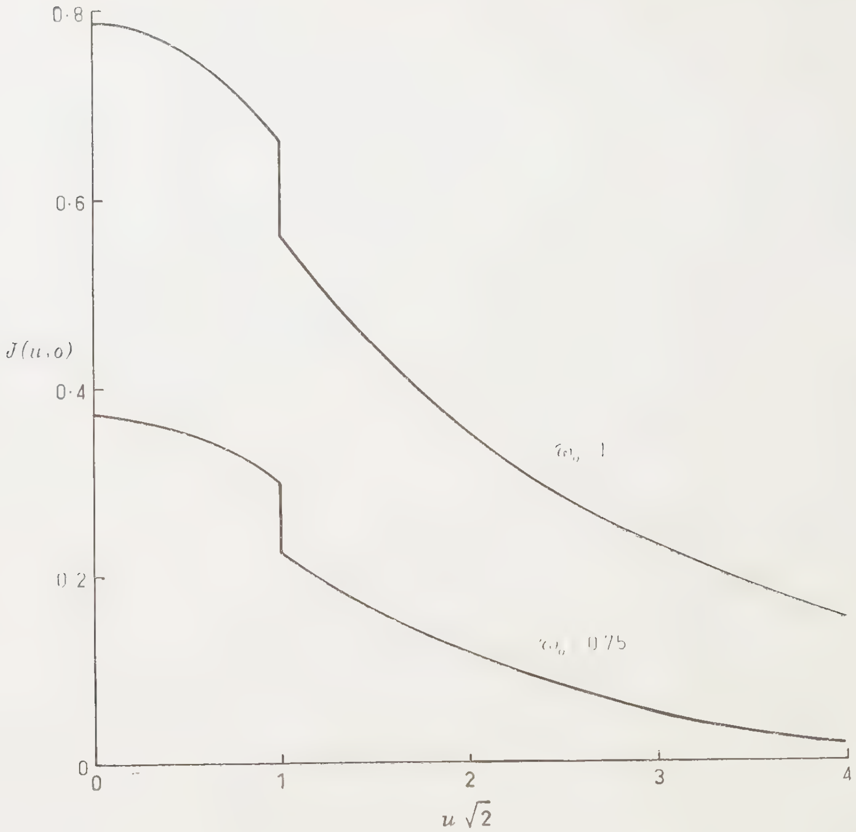

Fig. 3. — The variation of intensity at the surface for $\varpi_0 = 1$ and 0.75 with an incident flux

$$F(u) = \begin{cases} 1, & |u|\sqrt{2} \leqslant 1 \\ 0, & |u|\sqrt{2} > 1. \end{cases}$$

Rather than deal with the separate intensities I_i, we shall consider the mean intensity defined by the equation

$$J = \int I \, d\omega \sim \frac{2\pi}{3} \Sigma I_i.$$

At $\nu = 0$, the Fourier transform $\boldsymbol{J}$ of J may be written,

$$(13) \quad \boldsymbol{J}(\xi,0) = \frac{2\pi}{3} \Sigma \boldsymbol{I}_i = \frac{4\pi}{\varpi_0} [s(f)+s(g)+s(h)] - \psi(\xi)$$

which becomes, using (12),

$$\boldsymbol{J}(\xi,0) = \frac{6\pi(1-k_1)}{\varpi_0} \cdot \frac{\xi_2 \xi_3}{1-\eta^2} \psi(\xi) - \psi(\xi)$$

$$(14) \quad = \frac{6\pi(1-k_1)}{\varpi_0} \left(\frac{1-2\gamma-\eta^2}{1-\eta^2} \right) \psi(\xi) - \psi(\xi)$$

On inversion, (14) then gives

$$(15) \quad J(u,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \boldsymbol{J}(\xi,0) \mathbf{e}^{-i\xi u} d\xi$$

which may be evaluated numerically or by expanding (14) in a power series in ξ^2 and integrating term by term. In the latter case,

$$(16) \quad J(u,0) = -F(u) + \frac{12\pi}{(2\pi)^{3/2}\varpi_0} \sum_{n=0}^{\infty} \binom{1/2}{n} (-2\gamma)^n \left[1 - (1+\gamma)^{1/2-n}\right] \int_{-\infty}^{+\infty} \frac{\psi(\xi)\mathbf{e}^{-i\xi u} d\xi}{(1+\xi^2/2)^n}$$

where $\binom{1/2}{n}$ is the binomial coefficient.

As illustrations of the solution (15), cases have been considered corresponding to the input functions

(i) $F(u) = \mathbf{e}^{-|u|\sqrt{2}}$

(ii) $F(u) = \begin{cases} 1, & |u|\sqrt{2} \leqslant 1 \\ 0, & |u|\sqrt{2} > 1 \end{cases}$

and results for $\varpi_0 = 1$ and 0.75 are shown in figures 2 and 3. As the series (16) converges rather slowly, the results were obtained by numerical integration. The discontinuities in figure 3 are results of the approximation.

The intensity inside the medium follows from the

condition that the solution for $v > 0$ must fit smoothly onto that for $v = 0$. It follows therefore that

$$\mathbf{J}(\xi, v) = \mathbf{J}(\xi, 0)\, e^{-k_1 v} \tag{17}$$

and $J(u, v)$ is then found by inversion of (17).

4. **The reflectance.** — An indication of the accuracy of the above may be obtained on comparing the reflectance found on this approximation with the exact value obtained by GIOVANELLI [3] from CHANDRASEKHAR's results.

The outward flux of radiation per unit area F_0 is given, in the limit $\varepsilon \rightarrow 0$, by

$$F_0 = \frac{2\pi}{3} I_1 .$$

This may be written, in terms of the FOURIER transforms,

$$\mathbf{F}_0 = \frac{2\pi}{3} \mathbf{I}_1 = \frac{2\pi}{3} \frac{s(f)}{1 + k_1}$$

which, from (12) becomes

$$\mathbf{F}_0 = \left(\frac{1 - k_1}{1 + k_1}\right) \psi(\xi) . \tag{18}$$

The total energy passing outwards per second across an infinite strip of the surface of unit width and with its axis in the x direction is given by

$$\int_{-\infty}^{+\infty} F_0 \, du = \tag{19}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} \left(\frac{1 - k_1}{1 + k_1}\right) \psi(\xi)\, e^{-i\xi u}\, d\xi.$$

Similarly the total energy per second passing into the medium in a similar strip is

$$\int_{-\infty}^{+\infty} F \, du = \tag{20}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} \psi(\xi)\, e^{-i\xi u}\, d\xi.$$

Using FOURIER's integral theorem, i.e. that

$$\int_{-\infty}^{+\infty} d\chi \int_{-\infty}^{+\infty} f(\eta)\, e^{i\eta\chi}\, d\eta = 2\pi f(0),$$

the integrals (19) and (20) may be evaluated and the reflectance R obtained as the ratio

$$R = \frac{\int_{-\infty}^{+\infty} F_0 \, du}{\int_{-\infty}^{+\infty} F \, du} = \frac{1 - k_1(\xi = 0)}{1 + k_1(\xi = 0)} = \tag{21}$$
$$\frac{\sqrt{1 - \frac{2\varpi_0}{3}} - \sqrt{1 - \varpi_0}}{\sqrt{1 - \frac{2\varpi_0}{3}} + \sqrt{1 - \varpi_0}} .$$

Values of R computed from (21) are compared with exact values in Table 1. The agreement, while not perfect, indicates that the accuracy of this approximation is probably adequate for many purposes.

TABLE 1

Reflectance as a function of ϖ_0

ϖ_0	Equ (21)	Exact value *
0	0.0000	0.0000
0.2	0.0200	0.0352
0.4	0.0501	0.0834
0.6	0.1010	0.1554
0.8	0.2087	0.2853
0.9	0.3333	0.4150
0.95	0.4606	0.5356
0.99	0.7072	0.7528
0.999	0.8962	0.9128
1.000	1.0000	1.0000

* As given by GIOVANELLI (1955).

5. **Discussion.** — The case of normally incident radiation treated here in a first approximation is probably the simplest which could be chosen. For angles of incidence other than normal and for finer divisions of the angular coordinate μ, the above method yields very complex integrals which suggest that a FOURIER transform method of solution would be of little use in extending the approximation.

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Interim Report to the Commission Internationale de l'Eclairage, Zurich, 1955, on the National Physical Laboratory's investigation of colour-matching (1955)

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with an Appendix by W. S. Stiles and J. M. Burch
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Introduction. — The main reasons for re-examining the C. I. E. standard photometric and colorimetric data for the average eye were :

(a) doubts about the correctness of the standard luminous efficiency function V_λ in the ends of the spectrum, particularly the blue end, (b) small discrepancies between the colour differences of certain titanium pigments, as observed and as computed from the C. I. E. data, (c) the view of some workers in colour that colour measurements for a matching field larger than that used in determining the C. I. E. data might be more suitable as a basis for the standard system. To these reasons, may be added the remark that the method of determining the distribution coefficients, $\bar{x}, \bar{y}, \bar{z}$ by combining measurements of the unit coordinates for the spectral colours and a standard white, with measurements of the V_λ function, is not an ideal procedure, when we consider that $\bar{x}$, $\bar{y}$, $\bar{z}$ are directly measurable without introducing any heterochromatic brightness-matching. This is not a new point of view. GUILD expressed a similar opinion in 1931 when the C. I. E. colorimetric data were first agreed. He considered however that the method adopted was the best that could be done at the time.

The N. P. L. investigation of colour-matching has aimed at a direct measurement of the distribution coefficients, entirely independent of heterochromatic brightness-matching. Both small and large matching fields are being studied.

Knowing the distribution coefficients, the unit coordinates are immediately calculable. But the V_λ function is not completely determined without some additional measurements involving heterochromatic photometry. Certain additional measurements of this kind have been made. It will be clearer if the question of the V_λ function is deferred until after the work on the distribution coefficients has been summarised.

Because of delays for various reasons, it was not until August 1954 that the apparatus specially constructed for this work was completed and calibrated. Since then a considerable number of measurements have been made but I must emphasise that the mean results about to be considered are preliminary ones obtained for a pilot group of 10 subjects. Several interesting if difficult issues are raised by the results for the pilot group, on which it has been necessary to make special measurements — measurements not yet completed. In the light of all the preliminary work and of the discussions at the present meeting, the final conditions for measurements on a much larger group of 50 or more subjects will be fixed. These it is hoped might be completed in about one year and could form a contribution to a revised set of standard data which the Commission might later adopt by a postal vote, perhaps after an intersessional meeting of colorimetric specialists.

The N. P. L. trichromator. — A full description of the N. P. L. apparatus for this work was given recently (" *Thomas Young Oration* ", 25th February, 1955), but is not yet available in print. The main features are as follows. Three double " monochromators " of VAN CITTERT type with cancelling dispersion are mounted vertically one above the other. In the central spectrum of the middle tier, a single displaceable slit selects the monochromatic test colour which illuminates one half of the matching field. Three fixed slits in the upper tier select the monochromatic mixture primaries. These after recombination in the second stage of the monochromator provide a mixture which illuminates the other half of the matching field. The intensities of the primaries are independently variable by optical wedges inserted immediately after the slits in the central spectrum. The lower tier provides a similar mixture in variable amounts of three primaries — of the same wavelengths as the mixture primaries — which is added to the test colour and desaturates the latter. The exit slits of all three tiers are imaged at a common point in space and the subject's eye is brought into position so that the pupil centre coincides with this point and the whole of the light in the slit images — which all lie within a square area of about 2 mm side — is collected by the pupil. The subject sees the fields by the method of Maxwellian view. The photometric cubes are so made that the mixture (or comparison) beam from the top tier not only supplies the mixture half of the field but also provides a surround which extends the total area of illuminated field to a diameter of 14°. In order to establish the psychological pattern of a bipartite field distinct from a surround, a circular, " grey " demar-

cation line is introduced in the mixture beam. Cubes for two bipartite matching fields of diameters 2° and 10° respectively with horizontal dividing line are used, two cubes for each field size being provided so that the test field can be arranged to occupy either the upper or the lower half-field. The controls of the mixture wedges (and of the desaturation wedges) are so arranged that the intensity of the mixture can be varied without altering the proportions of the components, or alternatively the components can be varied independently, or, finally, any two components can be varied keeping their relative proportions the same.

The intensities in energy units of the test colour and of the primary stimuli were measured by means of a calibrated rubidium-on-silver photocell, permanently installed beyond the eye point of the apparatus so that frequent checks and corrections of the energy intensities emerging from the instrument could be made.

The three primary stimuli used for most measurements were as follows :

	Wave-number of centre	Wavelength λ	Overall band width
Red primary	15,418 cm^{-1}	648.6 mμ	555/23.1 cm^{-1}/mμ
Green primary	18,997	526.4	512/14.2
Blue primary	22,456	445.3	601/11.9

The intensity level at which colour-matches were made can best be expressed by giving the luminance in photopic trolands of the comparison half of the field, the luminous efficiencies of the mixture primaries being assigned their C. I. E. values. Evaluated in this way the levels used vary from one subject to another but the variation is considerable only in the blue end of the spectrum. Between 690 mμ and 490 mμ the troland value for a typical subject ranged between 400 and 1500. It dropped to about 180 at 460 mμ and remained about that value down to 410 mμ. Below 410 mμ and above 690 mμ the troland value dropped sharply to 10 at 392 mμ and 50 at 730 mμ, for a typical subject. The total band width of the test colour diminished steadily from 190 cm^{-1} (10 mμ) at 730 mμ to 96 cm^{-1} (2.2 mμ) at 476 mμ, and then rose to about 510 cm^{-1} (8.0 mμ) in the extreme blue 400-392 mμ.

Some desaturation is unavoidable when matching spectrum colours but the minimum amount is used if a single primary is added to the test colour for desaturation and the mixture field contains only the other two primaries. Initially this procedure was adopted for all test colours except those at and in the immediate neighbourhood of the primaries, where small amounts of two desaturating primaries must be added and all three primaries must be represented on the mixture side. It is of course necessary to see that the subject can " bracket " the match i.e. he must be able to set each of the three variable primaries (mixture or desaturation) so that the amount of the primary is (a) too great (b) too small, for a match to be possible by any adjustments of the other two variable primaries. Particular difficulties were experienced by almost all subjects for test stimuli in the range 460 mμ-420 mμ (approximately, subjects vary somewhat in their " worst " wavelengths). The trouble can best be described as a tendency to blue/green degeneracy as a result of which the range on the green (or blue) primary wedge over which " acceptable " matches can be made by adjustments of the blue (or green) and red primaries only, is unexpectedly large. The acceptable wedge range for the green primary is particularly wide and this introduces an instrumental error associated with the bracketing method of setting a logarithmic control (Compare Trezona, 1953, 1954). The error from this cause was reduced by adding green primary to both sides, so that the matches of test colours below 460 mμ were made in a less saturated colour field than would be possible theoretically with a blue primary of wavelength 445 mμ. In fact, the saturation corresponded approximately to the optimum obtainable with a blue primary at 460 mμ.

Mean data for pilot group. — A series of colour-matches through the spectrum gives the quantities of the three instrumental primaries in mixtures which match measured energy intensities of test colours of different wavelength. For every wavelength one of the quantities is of course negative, i.e. the corresponding primary is mixed with the test colour. By assuming the validity of the additive law for colour-matching, the intensities a_λ, b_λ, c_λ of the primaries to match unit energy intensity of each spectral colour are obtained. These data for each subject were immediately transformed to a set of reference primaries, which were spectral colours very close to the instrument primaries and located at wave numbers 15,500, 19,000, 22,500 respectively (645.2, 526.3, 444.4 mμ). As a result the colour-matching functions are obtained, the quantities x_λ, y_λ, z_λ, of the reference primaries to match unit energy intensity of the spectrum colours. The logarithms of the mean values of these quantities for the pilot group are given for both 2° and 10° matching fields in Table 2 of the summary of results appended. For comparison, the corresponding quantities for the C. I. E. standard colorimetric observer and for the modified observer suggested by Judd at the 1951 Meeting of the Commission, have been computed, and the values are given in Table 3 of the appendix. The transformations involved in such a comparison imply that the additive law in its broadest sense is valid for complete colour-matching. It will be recalled that Blottiau (1947) and Trezona (1953, 1954) have obtained evidence that the additive law is not accurately obeyed, and this point will be considered later. Meanwhile additivity is taken for granted.

There are very considerable individual variations in the colour-matching functions particularly in the extreme violet (400 mμ and below) where subjects may differ by some two log units. It must be remembered that at least some of the factors which cause the

wide spread at the ends of the spectrum in the V_λ curves of a group of subjects will also operate for the colour-matching functions.

Turning first to the mean 2° colour-matching functions, it may be said that for the " red " function the agreement with the C. I. E. values is fairly close except at wavelengths below 444 mμ. Here the agreement is better with Judd's modified C. I. E. values although even then the new values diverge from Judd's in the violet. The new " green " colour-matching function is materially higher than the C. I. E. and Judd functions at wavelengths between the green and blue primaries, while for the new " blue " colour-matching function the situation is reversed. Both the " green " and " blue " functions lie between the Judd and C. I. E. functions at wavelengths below the blue primary (444.4 mμ).

To analyse further these differences, a derivation has been made of the unit coordinates in a Wright system, that is, one in which the units of the red and green spectral primaries are adjusted to make equal the red and green components of a mixture matching a spectral-yellow (17 250 cm^{-1}, 579.7 mμ), while the unit of the blue spectral primary is adjusted to make equal the blue and green components of a mixture matching a spectral blue-green (20 500 cm^{-1}, 487.8 mμ). (See. Table 1 of the appendix). If the new colour-matching functions differed from the C. I. E. only by a factor dependent on wavelength in the same way for all three, " red " " green " and " blue " functions, the unit coordinates in the Wright system would be identical and the factor could be regarded as expressing a difference of transmission of a pigment layer in the eye, covering visual receptors of similar spectral sensitivities. In fact, the unit coordinates for the new 2° measurements are quite close to the C. I. E. values for wavelengths 700 to 490 mμ, but below 490 mμ certain differences are in evidence. These have to be considered in relation to the individual variations. For the " red " unit coordinate, a plot of individual deviations from the mean suggests that the difference of the mean and the C. I. E. values may be significant in the range 430 to 470 mμ (fig. 1). If so the new data differ from the C. I. E. in a way which cannot be attributed to differences of pigmentation of the groups of subjects concerned.

The information given by the unit coordinates on the Wright system can be supplemented by considering the adjustments of the units of the primaries which have to be made to transform the data to this system. In other words we consider the relative values of the original colour-matching functions at the normalising wavelengths. These values are :

		Mean 2°	C. I. E.	Mean 10°
17 250 cm^{-1} (579.7) mμ	$\dfrac{x_\lambda}{y_\lambda}$ = ...	3.40	3.56	4.11
20500 cm^{-1} (487.8) mμ	$\dfrac{z_\lambda}{y_\lambda}$ = ...	0.73	1.08	0.51

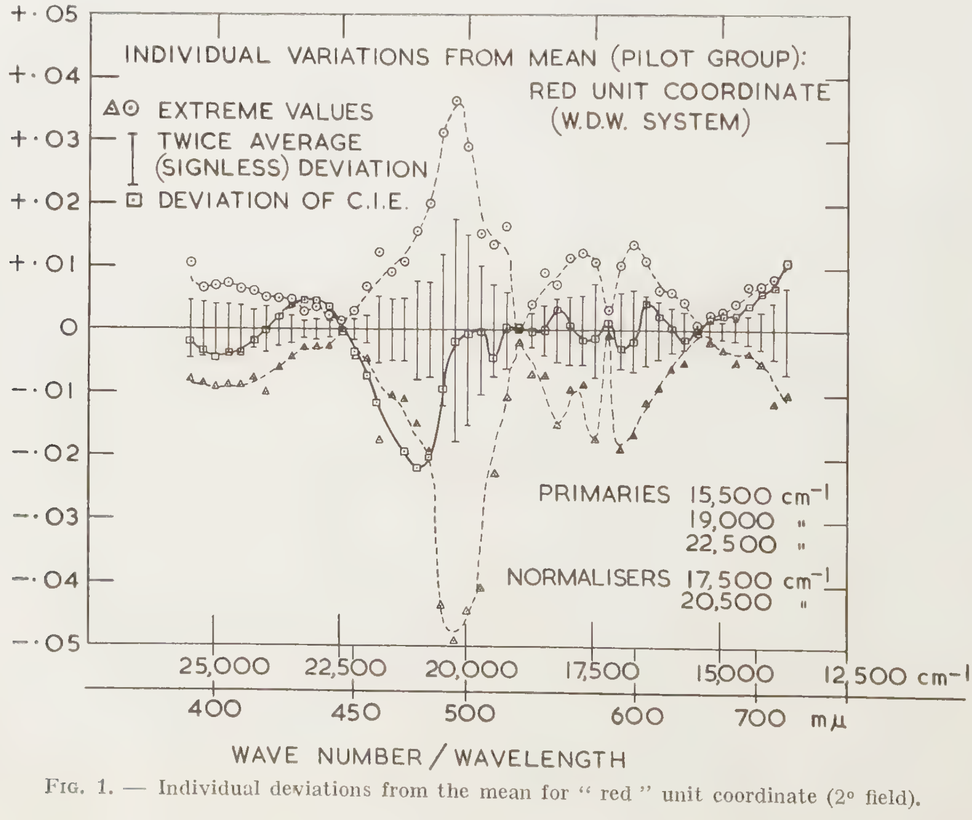

Fig. 1. — Individual deviations from the mean for " red " unit coordinate (2° field).

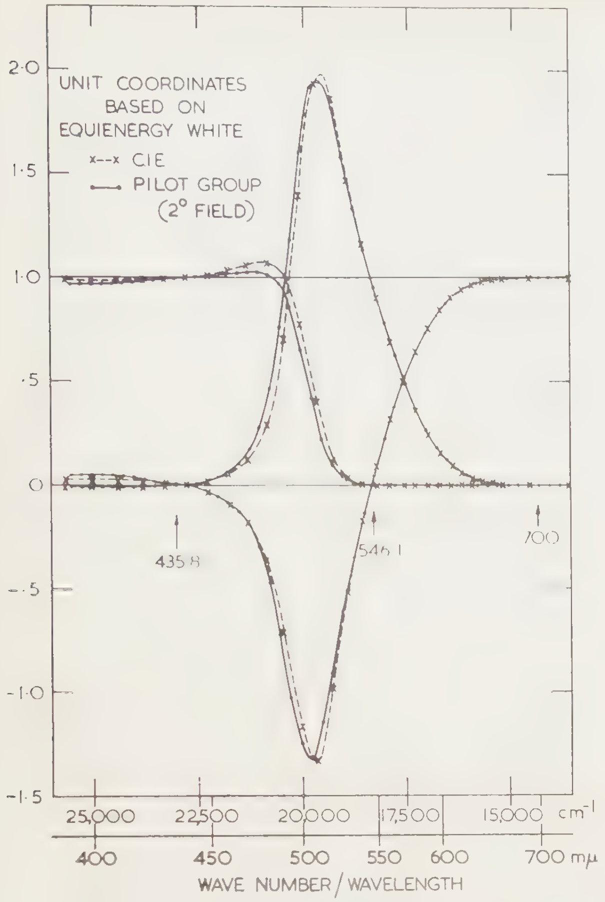

FIG. 2. — Mean unit coordinates for 2° field compared with C. I. E. r, g, b values.

At the yellow normalising wavelength the new 2° data and the C. I. E. give red to green ratios (x_λ/y_λ) which are satisfactorily close. At the blue-green normaliser, the blue to green ratio (z_λ/y_λ) is considerably lower for the new 2° data than for the C. I. E., a difference which is formally consistent with a smaller average density of macular pigment for the present subjects (It may be observed that the ratio is still further reduced in the new 10° results).

The modified C. I. E. observer proposed by JUDD in 1951 is sometimes referred to as an individual less pigmented than the original C. I. E. observer. But in so far as this description is correct, the pigmentation in question appears to be lens (or vitreous humour) and not macular pigmentation. JUDD's modification has the effect in the main of raising the values of the colour-matching functions in the violet : it represents an observer with rather more macular pigment than the C. I. E. observer. These conclusions follow from a consideration of the form of the spectral absorption curves of lens and macular pigments, and of the method used by JUDD to derive his modified functions.

The apparent difference in macular pigmentation between the new 2° data and the C. I. E. cannot in the main be attributed to the use of a possibly incorrect V_λ function in the derivation of the C. I. E. distribution coefficients. It represents a discrepancy with the GUILD-WRIGHT measurements themselves. This is shown by transforming the new data to the spectral primaries 700.0, 546.1, 435.8 mμ and deriving new unit coordinates, after adjusting the units of the primaries so that equal quantities are required in a mixture to match the equienergy white (wavelength basis). The calculation involves the integration of the colour-matching functions over the equienergy spectrum. The unit coordinates so obtained are directly comparable with the r, g, b, unit coordinates (coefficients trichromatiques) tabulated in the C. I. E. 1931 Colorimetry Resolutions (fig. 2). The major differences occur in the blue green where, for example, new unit coordinates at 490 mμ for the C. I. E. observer correspond approximately to those at 486.8 mμ for the new 2° data. The hue limen at 490 mμ is of the order of 1.0 mμ, so that the difference between the C. I. E. and the new 2° values is about three times what would be detectable by a particular subject.

For the present purpose a better way of assessing the discrepancy is to compare it with individual differences. In this comparison the individual data in terms of the original reference primaries — 15,500, 19.000, 22,500 cm⁻¹ — were used. All the subjects had made matches with a special " NPL bluish white " at the same time as their spectrum colour matches. For each subject the quantities of the primaries to match this special white were taken as the units for the primaries, and the unit coordinates for a selection of wavelengths were then evaluated. For the C. I. E. observer, the derivation of the unit coordinates on this basis entailed a calculation from the relative spectral energy distribution of the special bluish white. The special bluish white was in fact obtained by removing the slit in the central spectrum of the middle tier of the

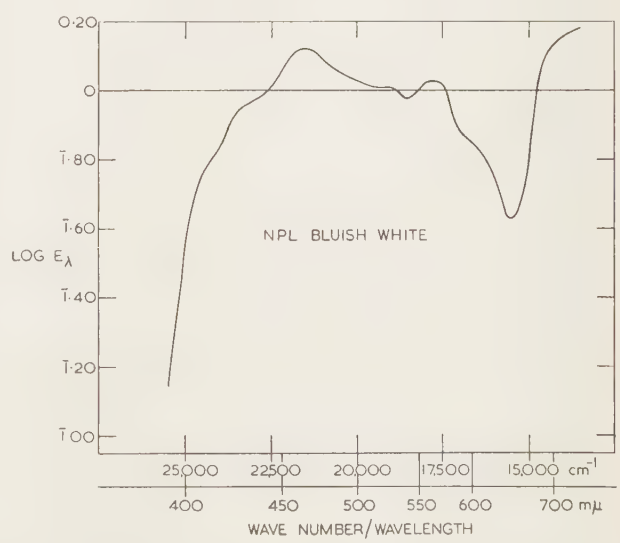

FIG. 3. — Spectral energy distribution for equal wavelength intervals of N. P. L. special " Bluish-white ".

trichromator so that reconstituted white light of measured spectral energy distribution filled the test half of the matching field. A neutral and a blue filter of known spectral transmissions were inserted to provide a white of suitable intensity and colour (fig. 3). In a rectangular chromaticity chart with the " red " and " green " unit coordinates as axes, the points corresponding to the matches of a particular spectrum colour made by the 10 subjects of the pilot group form a " cluster ", near the centre of which the calculated

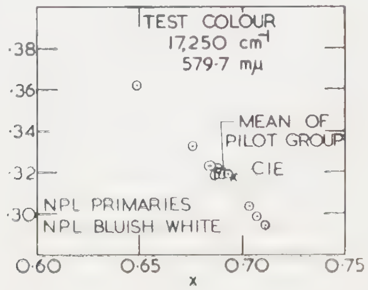

Fig. 4. — Chromaticity chart showing the location of 579.7 mμ for C. I. E. and for individual observers of pilot group (2° field).

point for the C. I. E. observer should be found if the agreement is good. This is the case for longer wavelengths as shown by the diagram for $\lambda = 579.7$ (17,250 cm^{-1}) (fig. 4). In the blue-green, however the C. I. E. point lies just on or just beyond the " edge " of the cluster. This is illustrated by the plot for $\lambda = 487.8$ (20,500 cm^{-1}) (fig. 5). If the points corresponding to the two groups of subjects on which the C. I. E. observer is based (the Guild and Wright groups) could be shown in this diagram they would form clusters overlapping the present one to some extent.

While it is possible that the average macular pigmentation for the present group of subjects happens to be less than that of the Guild-Wright groups by an amount corresponding to the above difference, a " sampling " error of this magnitude seems improbable when account is taken of the individual spread. This fact and the discrepancy in the unit coordinates on the Wright system, makes it necessary to look for other causes of the difference. But before doing so the large field results will be described.

Two special questions arise in large field matching : possible intrusion of rod vision, and apparent colour inhomogeneities in a small area of the field round about the direction of vision. At the field intensities used for the present work it seems improbable that rod vision is modifying materially the large field colour-match settings except in the extreme red end and possibly in the violet. I think we can understand why that should be from our knowledge of the spectral sensitivity and threshold sensitivity of the rod as compared with the cone mechanisms. I cannot enlarge on this point now and I shall merely indicate where in the mean results the effect of the rod mechanism is, we believe, making itself felt.

The colour inhomogeneities at the centre might be expected to make it difficult to do the actual colour-matching but they have in fact proved of little trouble as subjects soon learn to ignore the small area of the field where they occur. The inhomogeneities are no doubt the net result of local variations in retinal pigmentation, in the spectral sensitivities of the underlying end-organs and in the neural organization beyond them. They are visible only with certain rather special, highly saturated or dichroic stimuli and are, for example, hardly noticeable when matching the standard bluish white.

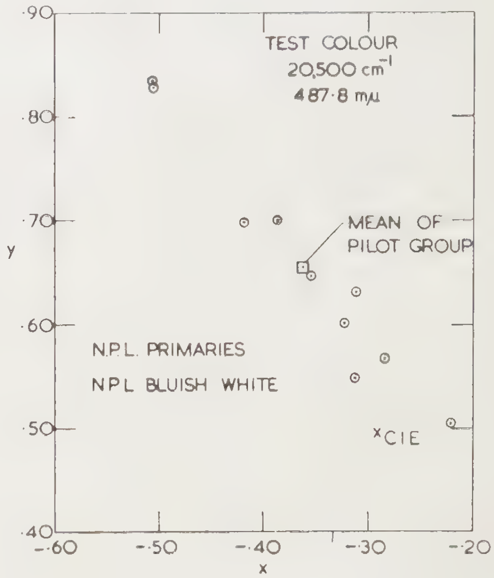

Fig. 5. — Chromaticity chart showing the location of 487.8 mμ for C. I. E. and for individual observers of pilot group (2° field).

The mean 10° colour-matching functions compared with the mean 2° functions show on the whole the kind of change to be expected from a reduction of the macular pigmentation. It appears however that the difference is not wholly explicable by assuming receptor systems with identical (or linearly related) spectral sensitivities under uniform pigment layers of different densities. This is apparent by a comparison of the unit coordinates in the Wright system. The negative " red " unit coordinate in the range 510 to 460 mμ is appreciably lower for the 10° data. The small drop in the " red " and the rises in the " blue " and negative " green " coordinates for deep reds beyond 700 mμ are probably the result of rod intrusion. From a plot of the ratios of the 10° to the 2° mean colour-matching functions it appears that for wavelengths below 550 mμ all three ratios have a maximum in the blue-green region recalling the absorption curve of macular pigment but the shapes of the curves are not identical as would be the case if the simple explanation men-

tioned earlier held good. There is also a difference in shape for wavelengths greater than about 580 mμ, which cannot be ascribed to a pigment and certainly not to the macular pigment that we know. At 700 mμ, the ratio curves for the blue and green colour-matching functions show sharp changes associated, it is believed, with rod vision.

To return to the 2° results there are certain differences between the present observational conditions and those used by WRIGHT and by GUILD which may be the reason for the discrepancies with the C. I. E. already noted. The principal differences in observational conditions with comments on their possible effects are given below. These comments are based on such auxiliary measurements as we have been able to make in the time available, but to deal satisfactorily with some of the possibilities more experimental work will be necessary.

Intensity Level of Matches. — From measurements made at 700, 580 and 490 mμ it appears unlikely that differences in intensity level have played an important part. There is some evidence that in the spectral region 420 to 470 mμ, a higher precision of colour-matching might have been obtained in the new work by making the observation at a lower intensity level.

Surround. — In the critical blue-green region cutting out the surround has little if any effect on the match settings. In the deep blue and violet there is some indication of an effect but this requires confirmation.

Degree of saturation. — It has already been noted that the addition of more green primary than theoretically necessary helped to stabilise the matches for wavelengths below 460 mμ. It does not appear however that, on the average, the resultant settings were much modified. Tests on the use of extra red desaturation in the blue-green region showed slight changes but not sufficient to account for the N. P. L./C. I. E. differences.

Pupil conditions. — The effective entrance pupil had a slightly larger area than that used by WRIGHT but was considerably small than GUILD's (about one seventh the area). From what is known of the dependence of colour on pupil of entry, the resultant hue difference in the critical regions in the blue-green and yellow would hardly exceed 0.5 mμ, and this would be too small to remove the major discrepancy. It might be making a small contribution however.

Different Choice of primaries. — The red primary is not materially different for all three investigations. GUILD's green primary had an effective wavelength at about 540 mμ compared with 530 (WRIGHT) and 526 (STILES and BURCH). GUILD and WRIGHT used a blue primary with effective wavelength at or near 460 mμ whereas a blue primary at 445 mμ was used in the present work. This last difference was thought to be the most likely source of the discrepancy, and for 3 subjects of the pilot group a series of 2° colour-matching measurements was made on the trichromator with the blue primary shifted from 445 mμ to 460 mμ. The observations were in the main confined to shorter wavelengths (526 mμ and below) where a difference would be most likely to appear. A similar series for the 10° field was made by another set of 3 subjects from the pilot group. The measurements were mainly confined to wavelengths below 526 mμ. The results were transformed to the same reference primaries 15,500, 19,000, 22,500 as before. The new colour-matching functions should be the same as the original ones for the subjects concerned if the additive law is valid. The mean curves — which with only 3 instead of 10 subjects are a good deal less smooth than the main data — show only a small and perhaps not significant difference. The slight effects are similar for the small and large fields. With the 460 blue primary the " blue " colour-matching function in its principal spectral region — 526 to the violet end — is a little less sharp, while the " green " and " red " functions in their main spectral regions — 460 mμ to 650 mμ and 526 to the red end respectively — are very little altered. This has the effect that the z_λ/y_λ ratio at 20,500 cm^{-1} is brought a little nearer the C. I. E. value. The effect on the z_λ/y_λ ratio at 20,500 cm^{-1} (487.8 mμ) of changing the blue instrument primary from 22,500 to 21,750 is shown by the following figures :

z_λ/y_λ (reference primaries 15,500, 19,000, 22,500)

" Blue " Instrument Primary	2°		10°		C.I.E.
	22,500	21,750	22,500	21,750	
	.72	.83	.47	.51	1.08
	means for 3 subjects (N^{os} : 1, 3, 7)		means for 3 subjects (N^{os} : 1, 7, 14)		

Although the ratio obtained with the 21,750 primary (2° field) moves somewhat nearer the C. I. E. value — and all 3 subjects show a change in the same direction — the gap is not much reduced. Another comparison that can be made is of the unit coordinates in a WRIGHT system (fig. 6). For the 2° case there is perhaps a slight systematic increase in the amount of desaturating " green " primary for test colours of wavelength below 444 mμ when changing the " blue " primary from 445 to 460 mμ. There is also some irregularity in the unit coordinates for the latter case at test colours round about 460 mμ. This may be connected with the rather different matching methods which

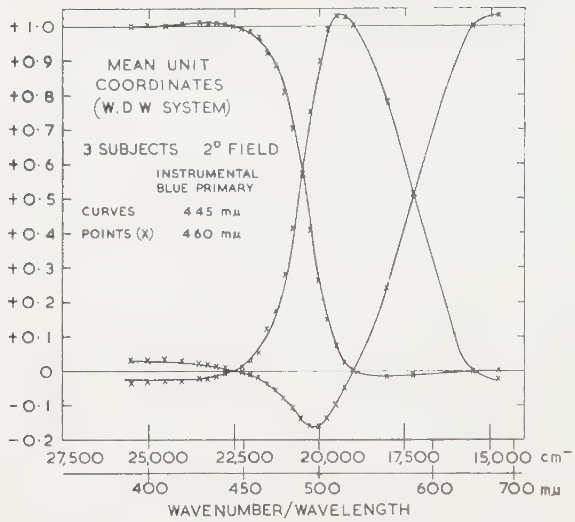

Fig. 6. — Effect of changing instrument " blue " primary from 445 mμ to 460 mμ on mean unit coordinates for three observers (W. D. W. system : 2° field).

must be used near the primary. For the 10° field, the unit coordinates obtained with the different instrumental " blue " primaries show even less difference than for the 2° field. An entirely independent examination of the effect on the unit coordinates of changing the instrumental " blue " primary from 445 to 460 mμ has been made for his own eye by Professor W. D. Wright using his well-known spectral colorimeter. He found that there was no appreciable difference in the unit coordinates for the two cases.

To sum up, we have been unsuccessful so far in establishing that one particular difference in observational conditions could account for the discrepancy between the Guild-Wright and the present 2° data. On the physical side, rechecks of the relative spectral sensitivity of our photocell, on which so much depends, have been very satisfactory and an independent determination of the colour of the bluish white light provided by the test tier of the apparatus, using a Donaldson 6-stimuli colorimeter, shows no appreciable difference from the colour calculated from the measured spectral energy distribution. It seems that the discrepancy may be the accumulation of small effects connected with observational conditions, coupled with an unexpectedly large difference in " effective " macular pigmentation of the groups of subjects concerned. Uncertainties on the latter account should be removed when the measurements are extended to a larger group of 50 or 60 subjects.

But we shall have to decide before making these further measurements precisely what observational conditions we will employ. It will be most valuable to us and to others making similar measurements to hear the views of delegates to this meeting of the C. I. E. on what the observational conditions should be if the results are to provide revised standard colour-matching functions. Perhaps the principal question to be considered is the desirable size of the matching field. I may perhaps indicate some of the points of measurement which will arise. With a large matching field the nature of the surround is probably not critical. With a small matching field on the other hand, a dark surround, a surround of the same luminance and colour as the matching field or, finally, a surround of the same luminance but white, may give some small systematic differences. The intensity level for large field measurements must be kept high to avoid rod intrusion, and there is in consequence less freedom in choosing " pupil of entry " conditions to approximate normal viewing. The choice of primaries, the degree of desaturation to be employed and whether it should be a primary stimulus or perhaps a standard white, are questions which arise with any field size. All the points mentioned have to be considered with the probable applications of the data in mind. A point of another kind is that a standardisation on small field data would enable the valuable Guild-Wright measurements to be incorporated — with a little difficulty, it is true — in new standard colour-matching functions.

Relative luminous efficiency function, V_λ. — The V_λ function was originally defined to give information about the relative energy intensities of spectrum colours which produce the same sensation of brightness. In the derivation of the C. I. E. distribution coefficients or colour-matching functions from the Guild-Wright measurements of the unit coordinates of the spectral colours and a standard white, a V_λ function was used. The validity of this derivation depends on the V_λ function being a linear combination of the colour-matching functions, and this would be true if the matching process used in deriving V_λ (a) obeyed the additive law, and (b) always assessed as equal fields in complete colour match. For this purpose it is not necessary for the V_λ function to give correct information about the brighnesses of different colours seen in juxtaposition. If the colour-matching functions are directly determined, the V_λ function ceases to be of primary interest in setting-up the standard colorimetric eye. However, it will probably be generally agreed that the V_λ function should continue to be a linear combination of the colour-matching functions, and once the latter have been fixed there will remain two independent constants, specifying the relative luminosities of the reference primaries, which can be assigned so as to provide a V_λ function best suited to the requirements of photometry.

In the N. P. L. work, the relative luminosities of the spectral reference primaries (15,500, 19,000, 22,500 cm^{-1}) were determined by direct comparison with a " white " mixture matching the NPL bluish white. The mean relative luminosity factors so obtained for the pilot group are compared with the C. I. E. values in the following Table :

	15,500	19,000	22,500 cm⁻¹
C. I. E.	.169	1.0	.036
Judd's Modified C. I. E.	.169	1.0	.052
NPL 2°	.288	1.0	.082
NPL 10°	.329	1.0	.123

As compared with the C. I. E. or Judd's modified C. I. E. $V\lambda$ values, our subjects assessed the luminosities of both blue and red primaries relative to the green at considerably higher values. Although luminosities measured by direct comparison in the presence of full colour differences are certainly not additive for most subjects, it is of interest to see what kind of $V\lambda$ function would result if we took the above directly measured factors in conjunction with the colour-matching functions of the same subjects. Alternatively we might take for the 2° case the Judd luminosity factors in conjunction with the same colour-matching functions, on the view that the flicker method of measuring $V\lambda$ on which Judd's data largely rest is not seriously non-additive. The drawback here is of course that the groups of subjects are different. The resulting $V\lambda$ functions are given in Table 4 of the Appendix. Both synthetic $V\lambda$ functions for the 2° case differ considerably from either the C. I. E. or the Judd $V\lambda$ functions.

A comparison of the synthetic $V\lambda$ functions, for the 2° and 10° fields based in each case on the colour-matching functions and luminosity factors for the appropriate field size is of interest. The effect of reduced macular pigment in the large field case is shown

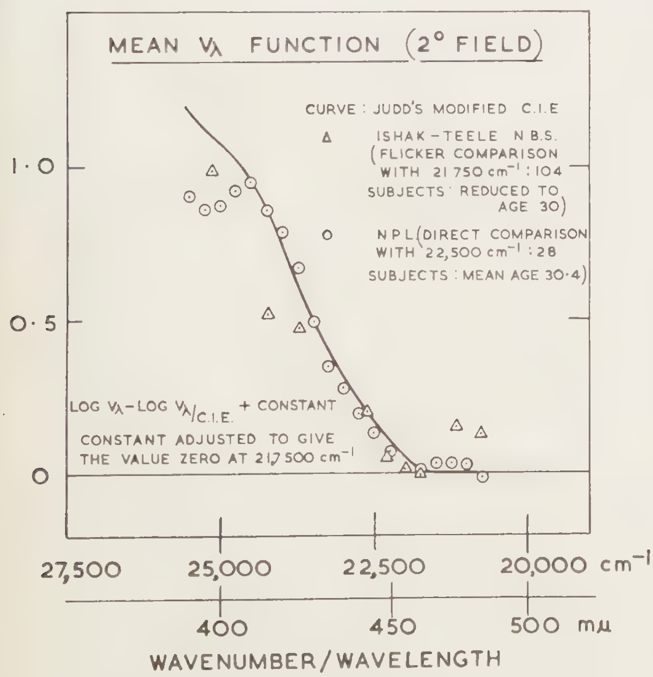

Fig. 7. — Log. relative luminious efficiency V_λ for the blue end of the spectrum: N. P. L. results for direct heterochromatic measurement with 28 subjects compared with C. I. E. Judd, and new 1 shak. Teele data (2° field).

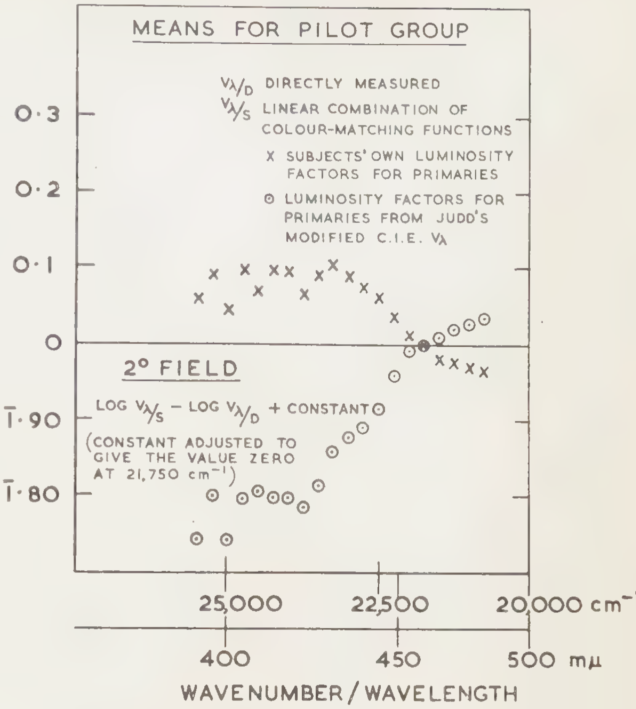

Fig. 8. — Comparison of directly measured V_λ function for pilot group in blue end of spectrum with two possible linear combinations of their colour-matching functions (2° field).

by the higher values for the 10° field, in the spectral region centred on 22,000 cm⁻¹ (455 mμ).

The N. P. L. work has also included a determination of the $V\lambda$ function for a 2° field in the blue end of the spectrum, by direct comparison with a monochromatic blue field of wavenumber 22,456 cm⁻¹ (the blue primary in fact). The mean results for a group of 28 subjects have been compared with the new Ishak-Teele measurements and with the Judd and C. I. E. $V\lambda$ functions in the accompanying graph (fig. 7). Both the new direct determinations of $V\lambda$ in the blue end of the spectrum confirm that the C. I. E. values are too low for wavelengths below 460 mμ (21,750 cm⁻¹). The relative luminosities for wavelengths in the blue end are in much better accord with Judd's modified $V\lambda$ function.

The 28 subjects used in the N. P. L. work on $V\lambda$ in the blue end included the pilot group of 10 subjects for whom the colour-matching functions had been measured. The question arises whether their mean direct measured $V\lambda$ curve in the blue end is a linear combination of their 2° colour-matching functions. Two linear combinations are compared with the directly measured $V\lambda$ values in figure 8. The combination using the subject's own directly measured luminosity factors (against white) is in fairly good agreement with $V\lambda$ direct, but the values are relatively low in the blue green. The combination using Judd's luminosity factors deviates more widely and

in the opposite sense. There is little doubt that an intermediate choice of the luminosity factors would give a rather close agreement. But the possibility of fitting with a linear combination of the colour-matching functions, a directly measured section of the V_λ curve, confined to wavelengths below about 480 mμ is not of particular significance. More interesting would be a similar test applied to the complete V_λ curve determined for the same subjects by some accepted procedure of heterochromatic photometry, flicker against white, for example. Sufficient measurements for such a comparison have not yet been made.

A final point worthy of note is that the mean direct measurements of V_λ in the blue end for the pilot group of 10 subjects are in fairly close agreement with the means for the expanded group of 28 subjects at least if the two wavelengths below 400 mμ are excluded. It does not appear from this that the pilot group has exceptional pigmentation characteristics in the blue end.

Acknowledgements. — In the early stages, Mr Robert Donaldson was associated with me in the work I have been describing. Since Mr Donaldson's tragic death, which we all regret so much, I have had the invaluable assistance of Dr J. M. Burch, without which it would have been impossible to reach as far as we have in the rather extensive programme we have set ourselves.

The work described above has been carried out as part of the research programme of the National Physical Laboratory, and this paper is published by permission of the Director of the Laboratory.

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APPENDIX

N. P. L. colour-matching investigation (1955) Mean results for pilot group of ten subjects

W. S. Stiles and J. M. Burch

Introduction. — The mean results are given for a pilot group of 10 subjects (ages, 20-53 ; mean age 31) who made measurements both for a 2° and 10° bipartite matching field with a surround extending to an outer diameter of 14°. The data are stated for reference stimuli of wave-numbers (wavelengths) 15,500 cm^{-1} (6,452 Å) 19,000 (5,263) and 22,500 (4,444), which are very close to the instrumental stimuli (15,418 (6,486), 18,997 (5,264) and 22,456 (4,453), approximately). The unit coordinates are stated in a W. D. W. system, that is to say, the units of the colour-matching functions are adjusted so that the " red " and " green " functions are equal at the normalising wave-number (wavelength) 17,250 (5,797) and the " green " and " blue " functions are equal at the normalising wave-number 20,500 (4,878). The C. I. E. and Judd's modified C. I. E. data — as given in the Secretariat Report of the Colorimetry Committee [C. I. E. Proc. Vol. 1, 7.11. (1951)] — have been transformed to the same reference stimuli and normalising wave-numbers and are included for comparison. Such a transformation assumes that the additive law, in its broadest sense, is valid for complete colour-matching.

On the view that the relative luminous efficiency function V_λ should be a linear combination of the colour-matching functions, it is necessary only to determine the relative luminous efficiencies of the reference primaries, to derive V_λ from the colour-matching functions. The results of this derivation are given for both 2° and 10° fields using the subject's own determinations of the relative luminous efficiencies of the reference stimuli by direct comparison with white under the same field conditions. In addition, for the 2° case, the subject's mean colour-matching functions have been combined using the relative luminous efficiencies of the reference stimuli taken from Judd's modified C. I. E. V_λ functions (C. I. E. 1951). The C. I. E. and Judd V_λ functions are also included for comparison.

In deriving the means for the group of 10 subjects for the following Tables, it is always the quantity itself — unit coordinate, colour-matching function, V_λ function — which is averaged, not the logarithm. In the case of the colour-matching functions and the V_λ function, in default of making both the mean quantity and the logarithm of the mean, the latter has been chosen as perhaps more convenient in giving a quick

survey of the results. An alternative method of deriving mean unit coordinates is to take as basis the mean colour-matching functions and to proceed as though these represented the properties of a single " mean " subject. The values of the unit coordinates for this mean subject differ slightly from the means of the unit coordinates derived independently for the individual subjects, which are the means quoted in the Table. The difference rarely exceeds 0.001 except in the violet where it reaches .003 for one wave number (10° field) and .004 for three wave numbers (2° field).

The calculations of the unit coordinates and the log colour-matching functions were made to at least one more figure than the quoted values. The latter are rounded to the third decimal place. For the unit coordinates this has the result that the sum for a given colour may differ from unity by ± 0.001 in some cases.

The final Table gives the mean results of a direct determination of the V_λ function in the blue end of the spectrum by the method of direct comparison with monochromatic blue of wave number 22,456 cm⁻¹. A larger group of 28 subjects, including the pilot group, was used and the means for the whole group and for the pilot group only are shown separately. The Table also includes for comparison the mean results of a recent direct determination of the V_λ function in the blue made by Ishak and Teele using the method of flicker photometry with a blue comparison light. All these data are for 2° field.

Table 1

Unit Coordinates (W. D. W. System)
Mean Results for Pilot Group and C. I. E. Values

		2° matching field			C. I. E.			10° matching field		
1/λ (cm⁻¹)	λ (Å)	15 500	19 000	22 500	15 500	19 000	22 500	15 500	19 000	22 500
13 750	7 273	1.025	— 0,030	0.005	1.035	— 0.037	0.001	0.999	— 0.023	0.024
14 000	7 143	1.029	— 0.032	0.003	1.036	— 0.037	0.001	1.012	— 0.028	0.016
14 250	7 018	1.030	— 0.032	0.002	1.036	— 0.037	0.001	1.022	— 0.031	0.009
14 500	6 897	1.031	— 0.033	0.002	1.035	— 0.036	0.001	1.029	— 0.034	0.005
14 750	6 780	1.029	— 0.031	0.002	1.031	— 0.032	0.001	1.023	— 0.028	0.005
15 000	6 667	1.023	— 0.026	0.003	1.025	— 0.026	0.001	1.017	— 0.024	0.007
15 250	6 557	1.015	— 0.016	0.001	1.016	— 0.017	0.000	1.012	— 0.016	0.004
15 500	6 452	1.000	0.000	000	1.000	000	000	1.000	000	000
15 750	6 349	0.973	0.028	— 0.001	0.972	0.029	— 0.001	0.976	0.027	— 0.003
16 000	6 250	0.932	0.070	— 0.002	0.932	0.069	— 0.002	0.938	0.069	— 0.007
16 250	6 154	0.875	0.128	— 0.003	0.877	0.126	— 0.003	0.878	0.129	— 0.007
16 500	6 061	0.797	0.207	— 0.004	0.801	0.204	— 0.005	0.804	0.203	— 0.007
16 750	5 970	0.708	0.299	— 0.006	0.706	0.302	— 0.008	0.711	0.297	— 0.009
17 000	5 882	0.604	0.405	— 0.009	0.601	0.410	— 0.011	0.607	0.403	— 0.010
17 250	5 797	0.505_5	0.505_5	— 0.011	0.507	0.507	— 0.013	0.506	0.506	— 0.011
17 500	5 714	0.401	0.613	— 0.013	0.399	0.617	— 0.016	0.412	0.601	— 0.013
17 750	5 634	0.314	0.702	— 0.016	0.313	0.705	— 0.018	0.317	0.698	— 0.016
18 000	5 556	0.236	0.781	— 0.017	0.236	0.783	— 0.019	0.239	0.777	— 0.016
18 250	5 479	0.166	0.851	— 0.017	0.170	0.849	— 0.019	0.170	0.847	— 0.017
18 500	5 405	0.109	0.906	— 0.016	0.109	0.907	— 0.016	0.108	0.907	— 0.015
18 750	5 333	0.054	0.955	— 0.010	0.054	0.956	— 0.011	0.052	0.958	— 0.011
19 000	5 263	000	1.001	— 0.001	000	1.000	000	0.002	0.999	000
19 250	5 195	— 0.052	1.031	0.002	— 0.052	1.031	0.021	— 0.048	1.022	0.026
19 500	5 128	— 0.094	1.027	0.067	— 0.099	1.031	0.068	— 0.088	1.024	0.064
19 750	5 063	— 0.135	0.987	0.147	— 0.135	0.986	0.149	— 0.122	0.984	0.138
20 000	5 000	— 0.156	0.890	0.266	— 0.156	0.893	0.263	— 0.142	0.887	0.255
20 250	4 938	— 0.161	0.745	0.416	— 0.163	0.751	0.412	— 0.138	0.735	0.403
20 500	4 878	— 0.145	0.573	0.573	— 0.155	0.577	0.577	— 0.124	0.562	0.562
20 750	4 819	— 0.112	0.405	0.707	— 0.132	0.409	0.724	— 0.096	0.390	0.705
21 000	4 762	— 0.083	0.270	0.812	— 0.104	0.271	0.834	— 0.068	0.256	0.812
21 250	4 706	— 0.056	0.168	0.888	— 0.075	0.172	0.903	— 0.045	0.155	0.890
21 500	4 651	— 0.041	0.112	0.929	— 0.055	0.106	0.949	— 0.029	0.093	0.936
21 750	4 598	— 0.025	0.062	0.963	— 0.037	0.066	0.970	— 0.018	0.052	0.966
22 000	4 545	— 0.014	0.035	0.980	— 0.022	0.038	0.984	— 0.010	0.026	0.984
22 250	4 494	— 0.006	0.013	0.993	— 0.010	0.017	0.993	— 0.005	0.012	0.993
22 500	4 444	000	— 0.001	1.001	000	000	1.000	000	000	1.000
22 750	4 396	0.003	— 0.007	1.003	0.007	— 0.012	1.001	0.004	— 0.008	1.005
23 000	4 348	0.008	— 0.013	1.005	0.012	— 0.022	1.001	0.007	— 0.014	1.007
23 250	4 301	0.012	— 0.018	1.006	0.016	— 0.029	1.012	0.010	— 0.018	1.008
23 500	4 255	0.015	— 0.021	1.006	0.019	— 0.033	1.014	0.014	— 0.021	1.007
23 750	4 211	0.020	— 0.022	1.002	0.022	— 0.037	1.015	0.018	— 0.024	1.007
24 000	4 167	0.023	— 0.022	0.998	0.023	— 0.039	1.016	0.020	— 0.026	1.006
24 250	4 124	0.026	— 0.023	0.996	0.024	— 0.040	1.015	0.022	— 0.029	1.006
24 500	4 082	0.029	— 0.018	0.989	0.025	— 0.040	1.015	0.023	— 0.027	1.003
24 750	4 040	0.030	— 0.024	0.994	0.026	— 0.041	1.015	0.024	— 0.029	1.004
25 000	4 000	0.031	— 0.024	0.993	0.027	— 0.041	1.014	0.025	— 0.027	1.002
25 250	3 960	0.030	— 0.027	0.997	0.027	— 0.041	1.014	0.025	— 0.029	1.004
25 500	3 922	0.030	— 0.026	0.996	0.028	— 0.041	1.014	0.025	— 0.030	1.005

TABLE 2

Log (Colour-matching Functions)
Mean Results for Pilot Group
(An " N " after the logarithm means that the function is negative)

		2° Matching Field			10° Matching Field		
1/λ (cm⁻¹)	λ (Å)	15 500	19 000	22 500	15 500	19 000	22 500
13 750	7 273	$\bar{3}.591$	$\bar{5}.529$ (N)	$\bar{6}.675$	$\bar{3}.593$	$\bar{5}.369$ (N)	$\bar{5}.076$
14 000	7 143	$\bar{2}.001$	$\bar{5}.972$ (N)	$\bar{6}.889$	$\bar{2}.017$	$\bar{5}.858$ (N)	$\bar{5}.305$
14 250	7 018	$\bar{2}.393$	$\bar{4}.369$ (N)	$\bar{5}.151$	$\bar{2}.416$	$\bar{4}.303$ (N)	$\bar{5}.455$
14 500	6 897	$\bar{2}.794$	$\bar{4}.777$ (N)	$\bar{5}.494$	$\bar{2}.820$	$\bar{4}.732$ (N)	$\bar{5}.592$
14 750	6 780	$\bar{1}.144$	$\bar{3}.096$ (N)	$\bar{5}.808$	$\bar{1}.171$	$\bar{3}.001$ (N)	$\bar{5}.975$
15 000	6 667	$\bar{1}.466$	$\bar{3}.346$ (N)	$\bar{4}.305$	$\bar{1}.489$	$\bar{3}.258$ (N)	$\bar{4}.421$
15 250	6 557	$\bar{1}.758$	$\bar{3}.435$ (N)	$\bar{4}.277$	$\bar{1}.177$	$\bar{3}.370$ (N)	$\bar{4}.405$
15 500	6 452	0.000	$\bar{6}.844$	$\bar{5}.576$ (N)	0.000	$\bar{5}.835$ (N)	$\bar{5}.650$
15 750	6 349	0.184	2.120	$\bar{4}.603$ (N)	0.165	2.004	$\bar{4}.747$ (N)
16 000	6 250	0.322	2.676	3.038 (N)	0.306	$\bar{2}.572$	3.317 (N)
16 250	6 154	0.410	$\bar{1}.050$	$\bar{3}.348$ (N)	0.402	$\bar{2}.967$	$\bar{3}.426$ (N)
16 500	6 061	0.450	$\bar{1}.338$	3.489 (N)	0.461	$\bar{1}.260$	3.524 (N)
16 750	5 970	0.450	$\bar{1}.549$	$\bar{3}.759$ (N)	0.471	$\bar{1}.488$	$\bar{3}.671$ (N)
17 000	5 882	0.412	$\bar{1}.711$	$\bar{3}.915$ (N)	0.451	$\bar{1}.668$	$\bar{3}.766$ (N)
17 250	5 797	0.348	1.820	2.038 (N)	0.387	$\bar{1}.783$	$\bar{3}.843$ (N)
17 500	5 714	0.255	$\bar{1}.911$	$\bar{2}.125$ (N)	0.320	$\bar{1}.880$	$\bar{3}.942$ (N)
17 750	5 634	0.135	$\bar{1}.955$	$\bar{2}.182$ (N)	0.204	$\bar{1}.942$	2.015 (N)
18 000	5 556	0.005	1.998	2.207 (N)	0.079	$\bar{1}.986$	$\bar{2}.019$ (N)
18 250	5 479	$\bar{1}.839$	0.020	$\bar{2}.194$ (N)	$\bar{1}.914$	0.007	$\bar{2}.013$ (N)
18 500	5 405	$\bar{1}.631$	0.022	$\bar{2}.134$ (N)	$\bar{1}.698$	0.018	$\bar{3}.967$ (N)
18 750	5 333	$\bar{1}.307$	0.022	$\bar{3}.897$ (N)	$\bar{1}.359$	0.016	$\bar{3}.777$ (N)
19 000	5 263	$\bar{4}.906$ (N)	0.000	$\bar{4}.911$ (N)	$\bar{3}.782$	0.000	$\bar{4}.074$ (N)
19 250	5 195	$\bar{1}.187$ (N)	$\bar{1}.954$	$\bar{2}.153$	$\bar{1}.248$ (N)	$\bar{1}.972$	$\bar{2}.090$
19 500	5 128	$\bar{1}.370$ (N)	$\bar{1}.880$	$\bar{2}.571$	$\bar{1}.468$ (N)	$\bar{1}.928$	$\bar{2}.434$
19 750	5 063	$\bar{1}.456$ (N)	$\bar{1}.794$	$\bar{2}.844$	$\bar{1}.573$ (N)	$\bar{1}.873$	$\bar{2}.730$
20 000	5 000	$\bar{1}.458$ (N)	$\bar{1}.688$	$\bar{1}.035$	$\bar{1}.619$ (N)	$\bar{1}.806$	$\bar{2}.979$
20 250	4 938	$\bar{1}.449$ (N)	$\bar{1}.585$	$\bar{1}.202$	$\bar{1}.618$ (N)	$\bar{1}.735$	$\bar{1}.188$
20 500	4 878	$\bar{1}.415$ (N)	$\bar{1}.483$	$\bar{1}.352$	$\bar{1}.620$ (N)	$\bar{1}.667$	$\bar{1}.378$
20 750	4 819	$\bar{1}.375$ (N)	$\bar{1}.402$	$\bar{1}.512$	$\bar{1}.576$ (N)	$\bar{1}.578$	$\bar{1}.546$
21 000	4 762	$\bar{1}.328$ (N)	$\bar{1}.314$	$\bar{1}.660$	$\bar{1}.518$ (N)	$\bar{1}.485$	$\bar{1}.697$
21 250	4 706	$\bar{1}.238$ (N)	$\bar{1}.186$	$\bar{1}.782$	$\bar{1}.429$ (N)	$\bar{1}.361$	$\bar{1}.831$
21 500	4 651	$\bar{1}.125$ (N)	$\bar{1}.037$	$\bar{1}.825$	$\bar{1}.302$ (N)	$\bar{1}.201$	$\bar{1}.912$
21 750	4 598	$\bar{2}.957$ (N)	$\bar{2}.829$	$\bar{1}.899$	$\bar{1}.140$ (N)	$\bar{2}.991$	$\bar{1}.968$
22 000	4 545	$\bar{2}.741$ (N)	$\bar{2}.598$	$\bar{1}.920$	$\bar{2}.903$ (N)	$\bar{2}.708$	0.000
22 250	4 494	$\bar{2}\cdot384$ (N)	$\bar{2}.196$	$\bar{1}.970$	$\bar{2}.540$ (N)	$\bar{2}.348$	$\bar{1}.994$
22 500	4 444	4.820	$\bar{3}.095$ (N)	0.000	$\bar{5}.817$ (N)	$\bar{3}.017$	0.000
22 750	4 396	$\bar{2}.209$	$\bar{3}.958$ (N)	$\bar{1}.981$	$\bar{2}.437$	$\bar{2}.181$ (N)	$\bar{1}.971$
23 000	4 348	$\bar{2}.507$	$\bar{2}.235$ (N)	$\bar{1}.949$	$\bar{2}.668$	$\bar{2}.349$ (N)	$\bar{1}.912$
23 250	4 301	$\bar{2}.630$	$\bar{2}.310$ (N)	$\bar{1}.900$	2.748	$\bar{2}.399$ (N)	$\bar{1}.836$
23 500	4 255	2.702	2.338 (N)	$\bar{1}.842$	2.789	$\bar{2}.376$ (N)	$\bar{1}.754$
23 750	4 211	$\bar{2}.705$	$\bar{2}.252$ (N)	$\bar{1}.745$	$\bar{2}.807$	$\bar{2}.346$ (N)	$\bar{1}.662$
24 000	4 167	$\bar{2}.640$	$\bar{2}.147$ (N)	$\bar{1}.606$	$\bar{2}.714$	2.218 (N)	$\bar{1}.512$
24 250	4 124	$\bar{2}.545$	$\bar{2}.002$ (N)	$\bar{1}.454$	$\bar{2}.613$	2.125 (N)	$\bar{1}.364$
24 500	4 082	$\bar{2}.392$	3.740 (N)	$\bar{1}.260$	2.456	$\bar{3}.916$ (N)	$\bar{1}.180$
24 750	4 040	$\bar{2}.206$	3.620 (N)	$\bar{1}.060$	$\bar{2}.251$	$\bar{3}.704$ (N)	$\bar{2}.950$
25 000	4 000	$\bar{3}.957$	$\bar{3}.333$ (N)	2.784	$\bar{2}.012$	$\bar{3}.433$ (N)	$\bar{2}.691$
25 250	3 960	$\bar{3}.708$	$\bar{3}.077$ (N)	$\bar{2}.548$	$\bar{3}.758$	$\bar{3}.190$ (N)	$\bar{2}.430$
25 500	3 922	$\bar{3}.405$	$\bar{4}.803$ (N)	$\bar{2}.242$	$\bar{3}.433$	$\bar{4}.853$ (N)	$\bar{2}.097$

TABLE 3

Log (Colour-matching Functions)
C. I. E. and JUDD's Modified C. I. E. Values
(An " N " after the logarithm means that the function is negative)

		C. I. E.			JUDD		
1/λ (cm⁻¹)	λ (Å)	15 500	19 000	22 500	15 500	19 000	22 500
					(approx. values : ± .015. Where there are gaps values are not materially different from C. I. E.).		
13 750	7 273	—	—	—	—	—	
14 000	7 143	$\bar{2}.082$	$\bar{4}.079$ (N)	$\bar{6}.614$	—	—	
14 250	7 018	$\bar{2}.448$	$\bar{4}.447$ (N)	$\bar{6}.946$	—	—	—
14 500	6 897	$\bar{2}.814$	$\bar{4}.803$ (N)	$\bar{5}.287$	—	$\bar{4}.65$ (N)	$\bar{5}.19$
14 750	6 780	$\bar{1}.174$	$\bar{3}.114$ (N)	$\bar{5}.591$	—	$\bar{3}.01$ (N)	$\bar{5}.52$
15 000	6 667	$\bar{1}.481$	$\bar{3}.335$ (N)	$\bar{5}.816$	—	$\bar{3}.16$ (N)	$\bar{5}.71$
15 250	6 557	$\bar{1}.769$	$\bar{3}.440$ (N)	$\bar{5}.929$	—	$\bar{3}.17$ (N)	$\bar{5}.79$
15 500	6 452	0.000	$\bar{5}.560$	$\bar{6}.969$	—	—	
15 750	6 349	0.180	$\bar{2}.106$	$\bar{4}.540$ (N)	—	$\bar{2}.21$	$\bar{4}.62$ (N)
16 000	6 250	0.319	$\bar{2}.639$	$\bar{3}.082$ (N)	—	—	$\bar{3}.14$ (N)
16 250	6 154	0.410	$\bar{1}.016$	$\bar{3}.464$ (N)	—	—	$\bar{3}.52$ (N)
16 500	6 061	0.453	$\bar{1}.306$	$\bar{3}.755$ (N)	—	—	$\bar{3}.80$ (N)
16 750	5 970	0.455	$\bar{1}.535$	$\bar{3}.984$ (N)	—	—	$\bar{2}.01$ (N)
17 000	5 882	0.425	$\bar{1}.707$	$\bar{2}.158$ (N)	—	—	$\bar{2}.21$ (N)
17 250	5 797	0.369	$\bar{1}.817$	$\bar{2}.267$ (N)	—	—	$\bar{2}.33$ (N)
17 500	5 714	0.285	$\bar{1}.922$	$\bar{2}.373$ (N)	—	—	$\bar{2}.42$ (N)
17 750	5 634	0.180	$\bar{1}.981$	$\bar{2}.425$ (N)	—	—	$\bar{2}.46$ (N)
18 000	5 556	0.050	0.018	$\bar{2}.443$ (N)	—	—	$\bar{2}.48$ (N)
18 250	5 479	$\bar{1}.889$	0.037	$\bar{2}.420$ (N)	—	—	$\bar{2}.45$ (N)
18 500	5 405	$\bar{1}.674$	0.041	$\bar{2}.319$ (N)	—	—	$\bar{2}.37$ (N)
18 750	5 333	$\bar{1}.335$	0.029	$\bar{2}.106$ (N)	—	—	$\bar{2}.16$ (N)
19 000	5 263	—	0.000	$\bar{6}.952$ (N)	—	—	—
19 250	5 195	$\bar{1}.204$ (N)	$\bar{1}.948$	$\bar{2}.287$	—	—	$\bar{2}.32$
19 500	5 128	$\bar{1}.398$ (N)	$\bar{1}.864$	$\bar{2}.714$	—	—	$\bar{2}.75$
19 750	5 063	$\bar{1}.449$ (N)	$\bar{1}.760$	$\bar{2}.973$	—	—	$\bar{1}.02$
20 000	5 000	$\bar{1}.437$ (N)	$\bar{1}.642$	$\bar{1}.145$	—	$\bar{1}.63$	$\bar{1}.19$
20 250	4 938	$\bar{1}.419$ (N)	$\bar{1}.531$	$\bar{1}.303$	—	$\bar{1}.52$	$\bar{1}.31$
20 500	4 878	$\bar{1}.404$ (N)	$\bar{1}.424$	$\bar{1}.457$	—	$\bar{1}.41$	$\bar{1}.49$
20 750	4 819	$\bar{1}.388$ (N)	$\bar{1}.326$	$\bar{1}.607$	—	$\bar{1}.27$	$\bar{1}.61$
21 000	4 762	$\bar{1}.355$ (N)	$\bar{1}.216$	$\bar{1}.738$	$\bar{1}.32$ (N)	$\bar{1}.15$	$\bar{1}.71$
21 250	4 706	$\bar{1}.278$ (N)	$\bar{1}.087$	$\bar{1}.840$	$\bar{1}.23$ (N)	$\bar{2}.98$	$\bar{1}.82$
21 500	4 651	$\bar{1}.211$ (N)	$\bar{2}.946$	$\bar{1}.930$	$\bar{1}.10$ (N)	$\bar{2}.81$	$\bar{1}.88$
21 750	4 598	$\bar{1}.069$ (N)	$\bar{2}.774$	$\bar{1}.972$	$\bar{2}.93$ (N)	$\bar{2}.66$	$\bar{1}.92$
22 000	4 545	$\bar{2}.854$ (N)	$\bar{2}.543$	$\bar{1}.992$	$\bar{2}.69$ (N)	$\bar{2}.42$	$\bar{1}.96$
22 250	4 494	$\bar{2}.505$ (N)	$\bar{2}.188$	$\bar{1}.998$	$\bar{2}.33$ (N)	$\bar{2}.02$	$\bar{1}.98$
22 500	4 444	—	($\bar{5}.591$)	0.000	—	—	0.00
22 750	4 396	$\bar{2}.345$	$\bar{2}.033$ (N)	$\bar{1}.988$	$\bar{2}.42$	$\bar{2}.03$ (N)	$\bar{1}.99$
23 000	4 348	$\bar{2}.560$	$\bar{2}.252$ (N)	$\bar{1}.955$	$\bar{2}.61$	$\bar{2}.29$ (N)	$\bar{1}.98$
23 250	4 301	$\bar{2}.619$	$\bar{2}.311$ (N)	$\bar{1}.893$	$\bar{2}.67$	$\bar{2}.36$ (N)	$\bar{1}.97$
23 500	4 255	$\bar{2}.576$	$\bar{2}.265$ (N)	$\bar{1}.780$	$\bar{2}.73$	$\bar{2}.35$ (N)	$\bar{1}.91$
23 750	4 211	$\bar{2}.458$	$\bar{2}.136$ (N)	$\bar{1}.608$	$\bar{2}.71$	$\bar{2}.32$ (N)	$\bar{1}.81$
24 000	4 167	$\bar{2}.278$	$\bar{3}.950$ (N)	$\bar{1}.400$	$\bar{2}.67$	$\bar{2}.27$ (N)	$\bar{1}.71$
24 250	4 124	$\bar{2}.096$	$\bar{3}.756$ (N)	$\bar{1}.196$	$\bar{2}.57$	$\bar{2}.19$ (N)	$\bar{1}.63$
24 500	4 082	$\bar{3}.869$	$\bar{3}.517$ (N)	$\bar{2}.950$	$\bar{2}.45$	$\bar{2}.07$ (N)	$\bar{1}.49$
24 750	4 040	$\bar{3}.682$	$\bar{3}.325$ (N)	$\bar{2}.755$	$\bar{2}.28$	$\bar{3}.94$ (N)	$\bar{1}.36$
25 000	4 000	$\bar{3}.522$	$\bar{3}.159$ (N)	$\bar{2}.585$	$\bar{2}.20$	$\bar{3}.79$ (N)	$\bar{1}.21$
25 250	3 960	$\bar{3}.309$	$\bar{4}.940$ (N)	$\bar{2}.364$	$\bar{2}.02$	$\bar{3}.63$ (N)	$\bar{1}.09$
25 500	3 922	$\bar{3}.021$	$\bar{4}.642$ (N)	$\bar{2}.063$	$\bar{3}.82$	$\bar{3}.44$ (N)	$\bar{2}.87$

Table 4

Log (Relative Luminous Efficiency, $V\lambda$)
Means for Pilot Group and C. I. E. and Judd's Modified C. I. E. Values

$1/\lambda$ (cm^{-1})	λ (Å)	C. I. E.	Judd (1951)	Linear combinations of colour-matching functions: 2° (Subjects' own direct comparison luminosity factors)	2° (Luminosity factors from Judd's 1951 curve)	10° (Subjects' own direct comparison luminosity factors)
13 750	7 273	$\bar{4}.801$	As for C. I. E.	$\bar{4}.929$	$\bar{4}.730$	$\bar{4}.959$
14 000	7 143	$\bar{3}.192$	»	$\bar{3}.344$	$\bar{3}.137$	$\bar{3}.381$
14 250	7 018	$\bar{3}.558$	»	$\bar{3}.736$	$\bar{3}.529$	$\bar{3}.780$
14 500	6 897	3.926	»	$\bar{2}.136$	3.929	$\bar{2}.182$
14 750	6 780	2.288	»	$\bar{2}.487$	$\bar{2}.281$	$\bar{2}.534$
15 000	6 667	2.602	»	$\bar{2}.811$	$\bar{2}.607$	$\bar{2}.854$
15 250	6 557	2.894	»	$\bar{1}.105$	$\underline{2}.907$	$\bar{1}.152$
15 500	6 452	$\bar{1}.137$	»	$\bar{1}.352$	1.161	$\bar{1}.366$
15 750	6 349	$\bar{1}.338$	»	$\bar{1}.550$	$\bar{1}.366$	$\bar{1}.542$
16 000	6 250	$\bar{1}.506$	»	$\bar{1}.707$	$\bar{1}.538$	$\bar{1}.701$
16 250	6 154	$\bar{1}.640$	»	$\bar{1}.821$	$\bar{1}.671$	$\bar{1}.816$
16 500	6 061	$\bar{1}.743$	»	$\bar{1}.906$	$\bar{1}.775$	$\bar{1}.904$
16 750	5 970	$\bar{1}.826$	»	$\bar{1}.957$	1.853	$\bar{1}.961$
17 000	5 882	$\bar{1}.891$	»	$\bar{1}.989$	$\bar{1}.912$	$\bar{1}.999$
17 250	5 797	$\bar{1}.941$	»	0.005	$\bar{1}.950$	0.008
17 500	5 714	$\bar{1}.974$	»	0.015	$\bar{1}.982$	0.021
17 750	5 634	1.993	»	0.002	$\bar{1}.988$	0.010
18 000	5 556	0.000	»	0.000	0.000	0.000
18 250	5 479	$\bar{1}.996$	»	$\bar{1}.986$	$\bar{1}.999$	$\bar{1}.977$
18 500	5 405	$\bar{1}.981$	»	$\bar{1}.959$	1.984	$\bar{1}.951$
18 750	5 333	$\bar{1}.954$	»	$\bar{1}.936$	$\bar{1}.970$	$\bar{1}.918$
19 000	5 263	$\bar{1}.910$	»	$\bar{1}.890$	$\bar{1}.934$	$\bar{1}.875$
19 250	5 195	$\bar{1}.845$	»	$\bar{1}.826$	$\bar{1}.876$	$\bar{1}.821$
19 500	5 128	$\bar{1}.749$	»	$\bar{1}.741$	$\bar{1}.792$	$\bar{1}.755$
19 750	5 063	$\bar{1}.636$	»	$\bar{1}.645$	$\bar{1}.696$	$\bar{1}.677$
20 000	5 000	$\bar{1}.509$	»	$\bar{1}.527$	$\bar{1}.582$	$\bar{1}.591$
20 250	4 938	$\bar{1}.391$	»	$\bar{1}.414$	$\bar{1}.473$	$\bar{1}.511$
20 500	4 878	$\bar{1}.277$	»	$\bar{1}.314$	$\bar{1}.369$	$\bar{1}.435$
20 750	4 819	$\bar{1}.177$	»	$\bar{1}.239$	$\bar{1}.296$	$\bar{1}.355$
21 000	4 762	$\bar{1}.074$	»	$\bar{1}.177$	$\bar{1}.223$	$\bar{1}.292$
21 250	4 706	$\bar{2}.970$	»	$\bar{1}.099$	$\bar{1}.130$	$\bar{1}.228$
21 500	4 651	$\bar{2}.871$	»	$\bar{1}.005$	$\bar{1}.021$	$\bar{1}.162$
21 750	4 598	$\bar{2}.774$	»	$\bar{2}.921$	$\bar{2}.910$	$\bar{1}.093$
			(Approx.)			
22 000	4 545	$\bar{2}.672$	$\bar{2}.72$	$\bar{2}.840$	$\bar{2}.808$	$\bar{1}.037$
22 250	4 494	$\bar{2}.568$	$\bar{2}.67$	$\bar{2}.811$	$\bar{2}.724$	$\bar{2}.995$
22 500	4 444	$\bar{2}.463$	$\bar{2}.63$	$\bar{2}.789$	$\bar{2}.654$	$\bar{2}.963$
22 750	4 396	$\bar{2}.351$	$\bar{2}.59$	$\bar{2}.758$	$\bar{2}.587$	$\bar{2}.904$
23 000	4 348	$\bar{2}.220$	$\bar{2}.51$	$\bar{2}.708$	$\bar{2}.489$	$\bar{2}.841$
23 250	4 301	$\bar{2}.069$	$\bar{2}.45$	$\bar{2}.653$	$\bar{2}.402$	$\bar{2}.765$
23 500	4 255	$\bar{3}.886$	$\bar{2}.36$	$\bar{2}.593$	$\bar{2}.313$	$\bar{2}.700$
23 750	4 211	$\bar{3}.661$	$\bar{2}.27$	$\bar{2}.528$	$\bar{2}.246$	$\bar{2}.621$
24 000	4 167	$\bar{3}.422$	$\bar{2}.16$	$\bar{2}.408$	$\bar{2}.110$	$\bar{2}.508$
24 250	4 124	$\bar{3}.211$	$\bar{2}.03$	$\bar{2}.279$	$\bar{3}.979$	$\bar{2}.348$
24 500	4 082	$\bar{4}.966$	$\bar{3}.92$	$\bar{2}.123$	$\bar{3}.859$	$\bar{2}.200$
24 750	4 040	$\bar{4}.770$	$\bar{3}.80$	$\bar{3}.907$	$\bar{3}.606$	$\bar{3}.994$
25 000	4 000	$\bar{1}.602$	$\bar{3}.65$	$\bar{3}.655$	$\bar{3}.354$	$\bar{3}.760$
25 250	3 960	4.384	3.47	$\bar{3}.415$	$\bar{3}.126$	$\bar{3}.500$
25 500	3 922	$\bar{4}.089$	$\bar{3}.29$	$\bar{3}.104$	$\bar{4}.797$	$\bar{3}.168$

TABLE 5

$\log_{10}$ (Relative Luminous Efficiency, $V\lambda$)
Direct Measurements for 2° Field
($V\lambda$ adjusted to have value unity at 21 750 cm^{-1})

$1/\lambda$ (cm^{-1})	λ (Å)	N. P. L. Pilot Group 10 Subjects	N. P. L. Extended Group 28 Subjects	ISHAK-TEELE 104 Subjects
20 750	4 819	0.357	0.382	—
20 790	4 810	—	—	0.509
21 000	4 762	0.288	0.321	—
21 187	4 720	—	—	0.368
21 250	4 706	0.203	0.222	—
21 500	4 651	0.104	0.120	—
21 750	4 598	0.000	0.000	—
21 786	4 590	—	—	$\bar{1}.995$
21 978	4 550	—	—	$\bar{1}.918$
22 000	4 545	$\bar{1}.909$	$\bar{1}.904$	—
22 250	4 494	$\bar{1}.857$	$\bar{1}.855$	—
22 371	4 470	—	—	$\bar{1}.796$
22 500	4 444	$\bar{1}.811$	1 813	—
22 624	4 420	—	—	$\bar{1}.830$
22 750	4 396	$\bar{1}.767$	$\bar{1}.765$	—
23 000	4 348	$\bar{1}.703$	$\bar{1}.718$	—
23 250	4 301	$\bar{1}.631$	$\bar{1}.637$	—
23 500	4 255	$\bar{1}.585$	$\bar{1}.600$	—
23 750	4 211	$\bar{1}.547$	$\bar{1}.549$	—
23 753	4 210	—	—	$\bar{1}.346$
24 000	4 167	$\bar{1}.397$	$\bar{1}.428$	—
24 213	4 130	—	—	$\bar{2}.982$
24 250	4 124	$\bar{1}.265$	$\bar{1}.287$	—
24 500	4 082	$\bar{1}.137$	$\bar{1}.136$	—
24 750	4 040	$\bar{2}.893$	$\bar{2}.910$	—
25 000	4 000	$\bar{2}.693$	$\bar{2}.693$	—
25 126	3 980	—	—	$\bar{2}.694$
25 250	3 960	$\bar{2}.407$	$\bar{2}.465$	—
25 500	3 922	$\bar{2}.128$	$\bar{2}.214$	—

Lettre à l'éditeur

Compensateur biréfringent à grand champ

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On emploie généralement le compensateur de Babinet en lumière parallèle pour la mesure d'une différence de marche produite par une lame cristalline. Si on l'utilise en lumière même peu convergente le champ est limité par les lignes isochromatiques qui sont des hyperboles : l'ordre d'interférences varie d'une façon continue dans tout le champ et introduit des erreurs dans les observations. Il est possible d'éviter cet inconvénient au moyen des combinaisons suivantes utilisant des lames taillées parallèlement à l'axe :

1) Si on remplace une lame cristalline d'épaisseur e par deux lames identiques d'épaisseur $\frac{e}{2}$ croisées en interposant une lame demi-onde, les biréfringences des deux lames s'ajoutent tandis que leurs variations qui sont de signes contraires se compensent en grande partie. Les lignes isochromatiques deviennent des cercles et le champ obtenu est 26 fois plus grand que celui d'une lame simple pour du quartz.

2) On obtient un résultat analogue en remplaçant une lame critalline par deux lames taillées l'une dans un cristal positif (quartz) l'autre taillée dans un cristal négatif (spath) le rapport des épaisseurs étant convenable.

3) Un troisième système consiste à superposer une lame taillée dans un premier cristal et 2 lames d'axes croisés taillées dans un deuxième cristal de signe contraire à celui du premier. Le choix des épaisseurs étant déterminé.

Ces différents systèmes ont été indiqués et utilisés par B. Lyot dans son filtre monochromatique polarisant. Ils peuvent être également employés pour réaliser un compensateur du type Babinet à grand champ.

En adoptant par exemple la première solution on a la figure 1. Les quatre lames A B C D sont quatre lames de quartz. Les lames A et D sont identiques et leur épaisseur est égale à l'épaisseur moyenne des lames prismatiques B et C. Ces 2 lames sont également identiques et leur ensemble forme une lame à faces parallèles. Les axes des lames C et D sont tournés de 90° par rapport à ceux des lames A et B. Entre A et B et entre C et D sont intercalées 2 lames demi-ondes E et F dont les axes sont à 45° des axes des lames en quartz. Il peut être intéressant dans certains cas de rendre les lames A et D également prismatiques

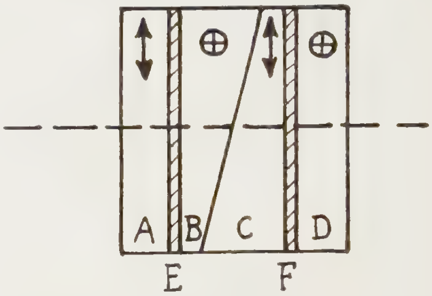

Fig. 1.

pour conserver l'égalité des épaisseurs A et B, C et D. L'ensemble des lames restant assimilable à une lame à faces parallèles.

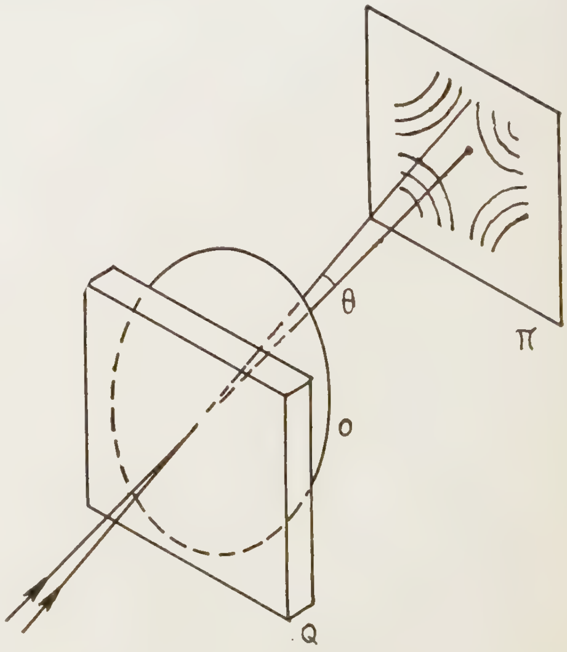

Fig. 2.

Le compensateur formé par ces lames montre des franges en lumière blanche comme dans le compensateur de BABINET ou le prisme de WOLLASTON et localisées également sur la face inclinée commune aux lames B et C. Avec un BABINET ou un WOLLASTON la différence de marche varie rapidement avec l'inclinaison des rayons. Sur la figure 2 le compensateur de BABINET Q (ou le prisme de WOLLASTON) est traversé par un faisceau convergent de lumière en formant l'image d'une source étroite sur Q. Un objectif O placé derrière Q montre dans son plan focal π les lignes isochromatiques des lames taillées parallèlement à l'axe. Ce sont des hyperboles et le champ angulaire θ correspondant à une différence de marche pratiquement constante c'est-à-dire à une teinte uniforme est petit. Si on remplace le BABINET ou le WOLLASTON par le nouveau compensateur on obtient un champ beaucoup plus grand et on peut utiliser un faisceau de lumière fortement convergent.

de SMITH dans lequel on a remplacé les deux prismes de WOLLASTON par deux compensateurs du nouveau type mais au lieu de disposer le compensateur dans le plan focal de l'objectif du microscope il est possible maintenant de le placer dans l'oculaire. L'image de la source lumineuse se forme en S au foyer du condenseur C. La préparation est en P et son image donnée par l'objectif O_1 du microscope, en P'. Un objectif auxiliaire O_2 donne du plan focal S', généralement non accessible, une image en S'' sur laquelle se trouve le compensateur Q_1. Le compensateur Q_1 possédant un grand champ il est possible de le placer en S'' où l'ouverture des faisceaux est plus grande qu'en S'. Ceci ne serait pas possible avec un BABINET ou un WOLLASTON dont les champs sont très petits. Quant à l'image P' l'objectif O_2 la transporte en P'' où elle est observée par l'oculaire O_c. Un deuxième compensateur Q_2 peut être placé en S conjugué de S'' à travers C, O_1 et O_2. La compensation est ainsi assurée et il est possible

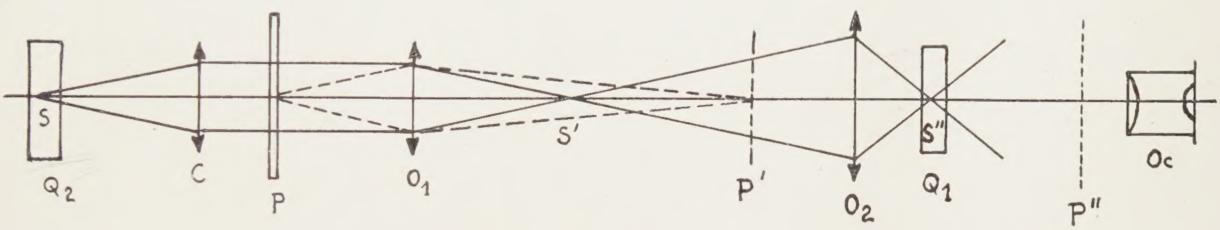

FIG. 3.

Le compensateur à grand champ peut être employé en microscopie interférentielle où grâce à ses propriétés il est possible de le placer dans l'oculaire du microscope.

Des montages interférentiels simples peuvent être réalisés pour l'étude des défauts d'homogénéité ou des défauts de poli dans le domaine macroscopique. De larges surfaces peuvent être étudiées.

La figure 3 représente l'adaptation du compensateur à grand champ au microscope. C'est le dispositif d'utiliser en S une source large. Si le faisceau n'est pas trop ouvert ce qui est en général le cas, un simple WOLLASTON suffira en S. Naturellement on place un polariseur avant Q_2 et un analyseur après Q_1.

La propriété d'autocompensation du prisme de WOLLASTON traversé deux fois, a été montrée par Léon et François LENOUVEL puis utilisée par NOMARSKI en microscopie. Cette propriété est conservée ici. L. et F. LENOUVEL ont indiqué un montage par autocollimation très lumineux qu'il est également

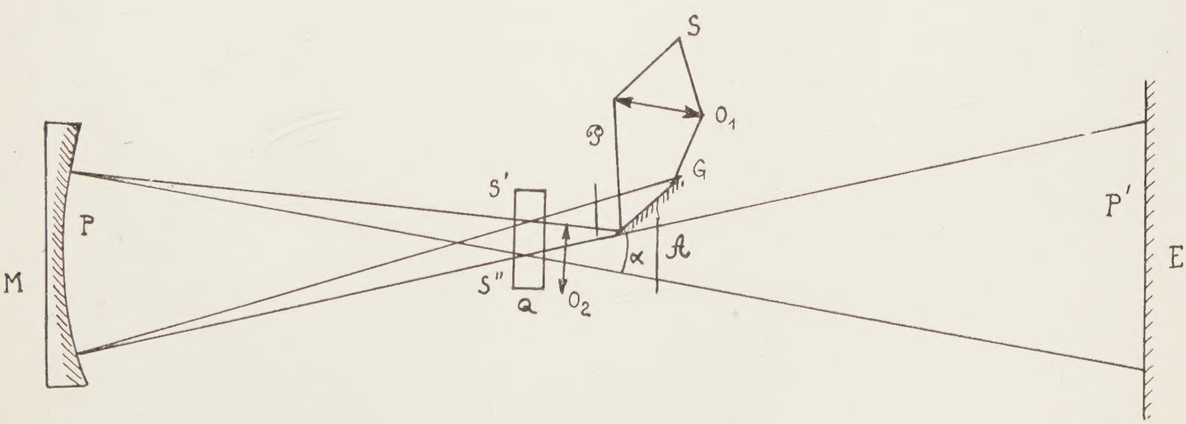

FIG. 4.

possible d'utiliser. L'objet transparent macroscopique est en P contre un miroir sphérique M (fig. 4). Au moyen d'un objectif O_1 et après réflexion sur un miroir G on forme en S' une image de la source S. L'image S' est au voisinage du centre du miroir M qui en donne une nouvelle image en S''. Le compensateur Q à grand champ est placé sur les images S' et S''. Un objectif O_2 donne de P une image en P' qui est projetée sur un écran E ou observée à l'oculaire. Le polariseur P est placé avant Q et l'analyseur A après Q sur le faisceau de retour. Grâce au champ considérable du compensateur il est possible d'utiliser un miroir M d'ouverture α importante et d'observer des objets P très étendus.

Un montage pour l'observation des objets transparents macroscopiques traversés une fois pourra être réalisé sur un principe analogue à celui de la figure 3 et avec deux objectifs seulement.

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